A first look into a formal and constructive approach for discrete geometry using nonstandard analysis.*

Laurent Fuchs[1], Gaëlle Largeteau-Skapin[1], Guy Wallet[2], Eric Andres[1], and Agathe Chollet[2]

[1] Laboratoire SIC,
Université de Poitiers,
BP 30179 86962 Futuroscope Chasseneuil cédex, France
{fuchs, glargeteau,andres}@sic.univ-poitiers.fr,
[2] Laboratoire LMA,
Université de La Rochelle,
Avenue Michel Crépeau 17042 La Rochelle cedex, France
Guy.Wallet@univ-lr.fr ; achollet@etudiant.univ-lr.fr.

Abstract. In this paper, we recall the origins of discrete analytical geometry developed by J-P. Reveillès [1] in the nonstandard model of the continuum based on integers proposed by Harthong and Reeb [2, 3]. We present some basis on constructive mathematics [4] and its link with programming [5, 6]. We show that a suitable version of this new model of the continuum partly fits with the constructive axiomatic of $\mathbb{R}$ proposed by Bridges [7]. The aim of this paper is to take a first look at a possible formal and constructive approach to discrete geometry. This would open the way to better algorithmic definition of discrete differential concepts.
Keywords: discrete geometry, nonstandard analysis, constructive mathematics.

1 Introduction

In the last twenty years Reveillès' approach to discrete geometry, namely Discrete Analytic Geometry (DAG), has become a very successful approach. DAG is based on the development of a powerful arithmetical framework which was originally founded on a special view of calculus: nonstandard analysis. The goal of this paper is to revisit some of these results and relate them to recent results on constructive mathematics [7].

Calculus, as initiated by Leibniz and Newton, deals with the concept of infinitesimals that are very small non-zero quantities. These infinitesimals have been used to define the notion of derivatives. However, even if powerful methods were developed by Leibniz and Newton, the notion of infinitesimal numbers wasn't well defined. These numbers, that are smaller than any positive number but still different from zero, didn't satisfy usual properties of real numbers. For

* Partially supported by the ANR program ANR-06-MDCA-008-05/FOGRIMMI.

example, any multiple of an infinitesimal number is still an infinitesimal number. This does not satisfy the Archimedean property: if x and y are two numbers such that $x < y$ then there exists an integer n such that $y < n.x$. Some paradoxes, such Zeno's paradox [8], also questioned the foundations of calculus. In the 19th century, this led to development of the, now, classical approach to calculus, based on the notion of limits defined on the continuum of real numbers. Later, in the mid of the 20th century an alternative approach, the nonstandard analysis, was proposed which adds infinitesimals and infinitely large numbers to the real numbers.

At the end of the eighties, at Strasbourg, Reeb and Harthong developed a nonstandard model of the continuum based on integers, the discrete-continuous Harthong-Reeb model [3]. This arithmetical description of the continuum was firstly tested on the numerical resolution of differential equations with integer numbers. We recall in this paper how the simple equation $y' = \alpha$, led Reveillès to his well known discrete analytical line definition and thus to the Discrete Analytic Geometry theory [1]. This study is mainly based on J. Harthong [2, 3], F. Diener and G. Reeb [9], M. Diener [10] and J-P. Reveillès and D. Richard [1] works. Since part of these works are in french, in our paper we have tried to summarize them to be self contained.

Interestingly, one of the difficulties that the development of discrete geometry faces today is the difficulty of correctly defining and using differential concepts. Our claim is that these difficulties come from a lack of theoretical foundations and effective methods (algorithms) to compute them. This is our motivation to reinvestigate the original nonstandard analysis point of view of discrete geometry. As Georges Reeb himself noted [9, 11], his model can be looked at from the constructivist (intuitionist) point of view. This has however never been really fully investigated although it represents a way to integrate the algorithmic point of view to the continuum theory [12]. In this paper, we take a first look into a formal and constructive approach for discrete geometry using nonstandard analysis.

For that, we show that a suitable version of the Harthong-Reeb model of the continuum partly fits with the constructive axiomatic of $\mathbb{R}$ proposed by Bridges [7]. Thus, this Harthong-Reeb model can be viewed as a constructive discrete-continuous model of the real line (called the Harthong-Reeb line). This is the first step in the project of giving theoretical and algorithmic definitions of discrete differential notions such as, for instance, the curvature of a curve.

2 Theoretical framework

In this part, we start with the origin of the Reveillès line to illustrate the strong link between the Harthong-Reeb approach to nonstandard analysis and discrete geometry. We explain the link between the integer nonstandard set and the real line $\mathbb{R}$. We also present an abstract on constructive mathematics and its link with programming.

2.1 Origin of the Reveillès discrete analytical line.

The Reveillès definition of a discrete naive straight line is classically given by [1]:

Definition 1. *A discrete analytical line of Reveillès is defined by*

$$D(a,b,\gamma,\tau) = \{(x,y) \in \mathbb{Z}^2, \gamma \leq ax - by < \gamma + \tau\}$$

where a, b, γ and τ are integers with $0 \leq a \leq b$, $b \neq 0$ and $\tau > 0$. In the case where $b = \tau$, this definition is equivalent to $y = \left\lfloor \frac{ax-\gamma}{b} \right\rfloor$.

In this definition, the *integer part* $\lfloor x \rfloor$ of a real number x is the largest integer less than or equal to x and the *fractional part* of x is the real number $\{x\} \in [0,1[$ such that $x = \lfloor x \rfloor + \{x\}$.

Originally, the definition of the discrete analytical line comes from the use of the Euler method (that numerically resolve ordinary differential equations) to the resolution of the differential equation $y'(x) = a$ such that $y(0) = b$. Solution of this equation is the straight line with equation $y(x) = ax + b$.

The Euler method gives the system

$$\left\{ x_0 = 0,\ y_0 = b,\ x_{n+1} = x_n + \frac{1}{\beta},\ y_{n+1} = y_n + \frac{a}{\beta} \right\}$$

where $\frac{1}{\beta}$ is an **infinitely small** integration step.

This system is arithmetized (i.e. transformed in an integer system) using an integer ω that can be viewed as a scale factor that allows to adjust the level of precision.

Everything works as if we move the coma in the usual real number representation: consider $\omega = 100$, then the real 3.12 becomes the integer 312 and no error is done on the two first digits after the coma. Hence, with an **infinitely large integer** ω, infinite precision is obtained. In practice, we work with an arithmetical analogous of the previous system:

$$\left\{ X_0 = 0,\ Y_0 = \lfloor \omega b \rfloor,\ X_{n+1} = X_n + \beta,\ Y_{n+1} = Y_n + \left\lfloor \frac{\lfloor \omega a \rfloor}{\beta} \right\rfloor \right\}$$

Note that, even if ω and β are independent, in practice it is useful to define $\omega = \beta^2$. For an integer T, we define $\mathbb{T}$ by the Euclidean division $T = \mathbb{T}\beta + r$. With this decomposition we have:

$$Y_{n+1} = \mathbb{Y}_{n+1}\beta + r_{n+1} = \mathbb{Y}_n\beta + \left\lfloor \frac{r_n + \left\lfloor \frac{\lfloor \omega a \rfloor}{\beta} \right\rfloor}{\beta} \right\rfloor \beta + \left\{ \frac{r_n + \left\lfloor \frac{\lfloor \omega a \rfloor}{\beta} \right\rfloor}{\beta} \right\}$$

$$= \left(\mathbb{Y}_n + \left\lfloor \frac{r_n + \left\lfloor \frac{\lfloor \omega a \rfloor}{\beta} \right\rfloor}{\beta} \right\rfloor \right) \beta + \left\{ \frac{r_n + \left\lfloor \frac{\lfloor \omega a \rfloor}{\beta} \right\rfloor}{\beta} \right\}$$

Since the decomposition in a base is unique, the following system is obtained:

$$\left\{ \begin{array}{l} \mathbb{X}_0 = 0,\ \mathbb{X}_{n+1} = \mathbb{X}_n + 1, \mathbb{Y}_0 = B,\ r_0 = \left\{ \frac{\lfloor \omega b \rfloor}{\beta} \right\}, \\ \mathbb{Y}_{n+1} = \mathbb{Y}_n + \left\lfloor \frac{r_n + A}{\beta} \right\rfloor,\ r_{n+1} = \left\{ \frac{r_n + A}{\beta} \right\}. \end{array} \right\}$$

where $A = \left\lfloor \frac{\lfloor \omega a \rfloor}{\beta} \right\rfloor$ and $B = \left\lfloor \frac{\lfloor \omega b \rfloor}{\beta} \right\rfloor$. This leads directly to the Reveillès algorithm [1] to draw the digitized straight line : $\mathbb{Y} = \frac{A}{\beta}\mathbb{X} + B$. The slope of this discrete line $\frac{A}{\beta}$ is an infinitely good approximation of a.

Note that to obtain the arithmetized Euler scheme, we have used an *"infinitely small"* value as integration step and an *"infinitely large"* value as scaling (precision) factor. Moreover, by choosing nonstandard values such that $\omega = \beta^2$ and adding the predicate "standard", denoted by st (which allows us to determine standard integers) to the usual Peano's axioms, we have the basis of the Harthong-Reeb model of nonstandard analysis (see below).

As usual, in order to deal with a model, it is useful to define a set of rules (axioms) that determine authorized expressions. Next section gives a minimal set of such rules that can be viewed as an approach of nonstandard analysis which is simpler and better adapted to our purpose than the usual theory [13, 14].

2.2 Bases of nonstandard analysis on $\mathbb{N}$ and $\mathbb{Z}$.

In this section we show that a continuum theory of the real line can be developed using only integers [2, 3, 9, 10, 1]. The ground idea of Harthong-Reeb model is that it suffices to introduce a scale over the usual set of integers to obtain a space that is both discrete and continuous. Nonstandard analysis is the paradigm that can be used to define such scales.

Even if axiomatic theories of nonstandard analysis, such as IST [13], are available, we present here axioms that are well suited for our purpose.

First we introduce a new predicate st over integer numbers: $st(x)$ "means" that the integer x is standard. This predicate is external to the classical integer theory and its meaning directly derives from the following axioms ANS1, ANS2, ANS3, ANS4 (and ANS5 which will be introduced later):

ANS1. *The number* 1 *is standard.*
ANS2. *The sum and the product of two standard numbers are standard.*
ANS3. *Nonstandard integer numbers exist.*
ANS4. *For all* $(x, y) \in \mathbb{Z}^2$ *such that* x *is standard and* $|y| \leq |x|$, *the number* y *is standard.*

For reading conveniences, we introduce the following notations:

- $\forall^{st} x\ F(x)$ is an abbreviation for $\forall x\ (st(x) \Rightarrow F(x))$ and can be read as "for all standard x, $F(x)$ stands".
- $\exists^{st} x\ F(x)$ is an abbreviation for $\exists x\ (st(x) \wedge F(x))$ and can be read as "exists a standard x such that $F(x)$".

Here we have to insist on the fact that these rules are added to every classical property (axioms or theorems) over integer numbers. Everything that was classically true remains true. We simply improve the language by a syntactic provision. These first rules imply that $\mathbb{N}$ is split into two classes, the class $\mathbb{N}_{st} := \{0, 1, \ldots\}$ of natural standard integers (closed by arithmetical operations), and the class of natural nonstandard integers (a nonstandard integer is bigger than every standard integer). These nonstandard integers are said *infinitely large*. These first axioms allow the development of an explicit and rigorous calculus on the different scales.

Let us add some technical but important remarks. A formula $\mathcal{P}$ is said *internal* if it does not bring in elements of the greatness scale. For example, the formula $x + 1 > x$ is internal. An internal formula is therefore a classical formula on numbers. In contrast, an *external* formula uses explicitly the greatness scale ; for example, $st(x)$ or $\forall^{st} x,\ y < x$ are external formulae. Since everything that was true remains true, for all internal formula $\mathcal{P}(x)$, we can build the set $P = \{x \in \mathbb{N}\ ;\ \mathcal{P}(x)\}$ which possesses the classic properties of subsets of $\mathbb{N}$; for example, if P is not empty and is bounded, then P possesses a bigger element which is not necessarily valid for an external property. For instance, if we consider the external property $st(x)$, the class $\mathbb{N}_{st} = \{x \in \mathbb{N}\ ;\ st(x)\}$ of standard integers is a non empty bounded part which cannot have a bigger element since $x + 1$ is standard for all standard x. A class of numbers defined by an external property which cannot be a set of numbers in the classical meaning is called *external set*. Hence, $\mathbb{N}_{st}$ is an external part of $\mathbb{N}$. Dealing with external sets that are not classical (internal) sets gives birth to a new process of demonstration called the *overspill principle*.

Proposition 1. ***(Overspill principle)*** *Let $\mathcal{P}(x)$ be an internal formula such that $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}_{st}$. Then, there exists an infinitely large $\nu \in \mathbb{N}$ such that $\mathcal{P}(m)$ is true for all integers m such that $0 \leq m \leq \nu$.*

Proof. The class $A = \{x \in \mathbb{N}\ ;\ \forall y \in [0, x]\ \mathcal{P}(y)\}$ is an internal set (i.e. a classical set) containing $\mathbb{N}_{st}$. Since $\mathbb{N}_{st}$ is an external set, the inclusion $\mathbb{N}_{st} \subset A$ is strict and leads to the result. □

In the same way, the application of an inductive reasoning on an external formula can be illegitimate. For example, number 0 is standard, $x + 1$ is standard for all standard x. Nevertheless not all integers are standard. To improve the power of our nonstandard tool, we have to add a special induction that fits with external formulae. In the following principle which is our last axiom, $\mathcal{P}$ denotes an internal or external formula:

ANS5. ***(External inductive defining principle):*** *We suppose that*

1. *there is $x_0 \in \mathbb{Z}^p$ such that $\mathcal{P}((x_0))$;*
2. *for all $n \in \mathbb{N}_{st}$ and all sequence $(x_k)_{0 \leq k \leq n}$ in $\mathbb{Z}^p$ such that $\mathcal{P}((x_k)_{0 \leq k \leq n})$ there is $x_{n+1} \in \mathbb{Z}^p$ such that $\mathcal{P}((x_k)_{0 \leq k \leq n+1})$.*

Therefore, there exists an internal sequence $(x_k)_{k\in\mathbb{N}}$ *in* $\mathbb{Z}^p$ *such that, for all* $n \in \mathbb{N}_{st}$*, we have* $\mathcal{P}((x_k)_{0\leq k\leq n})$.

This principle means that the sequence of values x_k for k standard can be prolonged in an infinite sequence $(x_k)_{k\in\mathbb{N}}$ defined for all integers. Saying that this sequence is internal means that this is a classical sequence of the number theory. Particularly, if $\mathcal{Q}(x)$ is an internal formula, then the class $\{k \in \mathbb{N} \; ; \; \mathcal{Q}(x_k)\}$ is an internal part of $\mathbb{N}$.

2.3 The system $\mathcal{A}_\omega$.

Now we are going to give the definition of the system $\mathcal{A}_\omega$. Introduced by M. Diener [10], this system is the formal version of the so-called Harthong-Reeb line. The underlying set depends on a parameter ω which is an infinitely large integer. In the next section (cf. section 3) we prove that this system can be viewed as a constructive model of the real line.

Accordingly to axiom ANS3, the construction starts by considering $\omega \in \mathbb{N}$ is an infinitely large (nonstandard) integer. We then introduce the set:

Definition 2. *The set* $\mathcal{A}_\omega$ *of the admissible integers considering the scale* ω *is defined by:* $\mathcal{A}_\omega = \{x \in \mathbb{Z} \; ; \; \exists^{st} n \in \mathbb{N} \; |x| < n\omega\}$.

The set $\mathcal{A}_\omega$ is an external set. Moreover, it is an additive sub-group of $\mathbb{Z}$. We provide $\mathcal{A}_\omega$ with the operations $+_\omega$ and $*_\omega$, the ω-scale equality, the ω-scale inequality relations (noted $=_\omega$ and $\neq_\omega$) and the order relation $>_\omega$:
We note $+, -, ., /, >$ the usual operations and order relation in $\mathbb{Z}$.

Definition 3. *Let* X *and* Y *be any elements of* $\mathcal{A}_\omega$.

- X *and* Y *are equal at the scale* ω *and we write* $X =_\omega Y$ *when* $\forall^{st} n \in \mathbb{N} \quad n|X - Y| \leq \omega$.
- Y *is strictly greater than* X *at the scale* ω *and we write* $Y >_\omega X$ *when* $\exists^{st} n \in \mathbb{N} \quad n\,(Y - X) \geq \omega$.
- X *is different from* Y *at the scale* ω *and we write* $X \neq_\omega Y$ *when* $(X >_\omega Y$ *or* $Y >_\omega X)$
- *The sum of* X *and* Y *at the scale* ω *is* $X +_\omega Y := X + Y$ *(like the usual sum). For this operation, the neutral element is* $0_\omega = 0$ *and the opposite of each element* $Z \in \mathcal{A}_\omega$ *is* $-_\omega Z := -Z$.
- *The product of* X *and* Y *at the scale* ω *is* $X \times_\omega Y := \lfloor \frac{X.Y}{\omega} \rfloor$ *(different from the usual one). The neutral element is* $1_\omega := \omega$*, and the inverse of each element* $Z \in \mathcal{A}_\omega$ *such that* $Z \neq_\omega 0_\omega$ *is* $Z^{(-1)_\omega} := \left\lfloor \frac{\omega^2}{Z} \right\rfloor$.

Let us give an informal description of $\mathcal{A}_\omega$. It is easy to see that $X = Y$ implies $X =_\omega Y$ but that the reverse is not true. It is a little less obvious to see that we have $\forall X \in \mathcal{A}_\omega, \; st(X)$ implies $X =_\omega 0$ but not the reverse. Indeed, for instance, $\lfloor\sqrt{\omega}\rfloor =_\omega 0$ (because $\forall^{st} n \in \mathbb{N}, n.\lfloor\sqrt{\omega}\rfloor < \omega$) but $\lfloor\sqrt{\omega}\rfloor$ isn't a standard integer or else ω would also be.

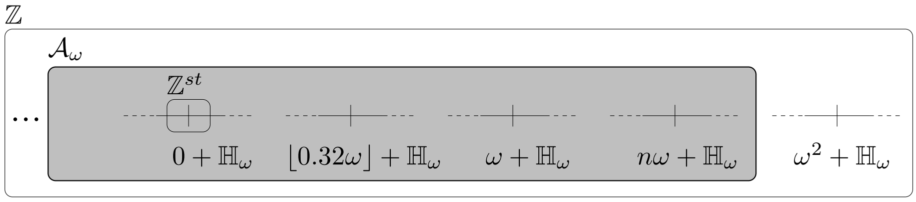

Fig. 1. The integer set $\mathbb{Z}$ and the set $\mathcal{A}_\omega$ (in grey).

Figure 1 illustrates how a representation of $\mathcal{A}_\omega$ could look like. Let us define the set $\mathbb{H}_\omega := \{X \in \mathcal{A}_\omega \ ; \ X =_\omega 0\}$. We have a strict inclusion of $\mathbb{Z}^{st}$ into $\mathbb{H}_\omega$. Let us consider the classical real value 0.32. In $\mathcal{A}_\omega$, the integer $\lfloor 0.32\omega \rfloor$ will be a representation of the classical real value 0.32 with an infinite precision. Of course, so does $\lfloor 0.32\omega \rfloor + 150$ or any integer x belonging to $\lfloor 0.32\omega \rfloor + \mathbb{H}_\omega$ since they all verify $x =_\omega \lfloor 0.32\omega \rfloor$. It is easy to see that $\lfloor 0.32\omega \rfloor$ is neither in $0 + \mathbb{H}_\omega$ nor in $\omega + \mathbb{H}_\omega$ (nor in $\lfloor 0.319\omega \rfloor + \mathbb{H}_\omega$ for that matter). The set $\lfloor 0.32\omega \rfloor + \mathbb{H}_\omega$ is sometimes called the halo of 0.32. The set $\mathcal{A}_\omega$ is in grey on the figure.
As we can see, $\mathcal{A}_\omega$ doesn't extend to $\omega^2 + \mathbb{H}_\omega$. The set $\mathcal{A}_\omega$ is a subset of $\mathbb{Z}$. It contains all integers from $-n\omega + \mathbb{H}_\omega$ to $n\omega + \mathbb{H}_\omega$ with n **standard**. Whereas the set $\mathbb{Z} \setminus \mathcal{A}_\omega$ contains all the integers $N\omega + \mathbb{H}_\omega$ with N **nonstandard**. Particularly $\mathcal{A}_\omega$ contains all $\lfloor k.\omega \rfloor + \mathbb{H}_\omega$ for k limited in $\mathbb{R}$ (i.e. $k \in \mathbb{R}$ such that $\exists^{\text{st}} n \in \mathbb{N}$ with $|k| \leq n$) and thus representations of all the classical real numbers.

2.4 Constructive mathematics, proofs and programs

In this section we will briefly introduce constructive mathematics and shortly draw the links with programming. For interested readers more details can be found in [7, 15, 16, 5, 17].

As explained by P. Martin-Löf in [5] :

> The difference between constructive mathematics and programming does not concern the primitive notions [...] they are essentially the same, but lies in the programmer's insistence that his programs be written in a formal notation [...] whereas, in constructive mathematics [...] the computational procedures are normally left implicit in the proofs [...].

Constructive mathematics has its origins, at the beginning of 19th century, with the criticisms of the formalist mathematical point of view developed by Hilbert which led to what we now call "classical mathematics". Brouwer was the most radical opponent to formal mathematics in which one can prove the existence of a mathematical object without providing a way (an algorithm) to construct it [18]. But his metaphysical approach to constructivism (intuitionism) was not successful. Around 1930, the first who tried to define an axiomatization of constructive mathematics was Arend Heyting, a student of Brouwer. In the mid of the fifties, he published a treaty [19] where intuitionism is presented to both mathematicians and logicians. From Heyting's work it became clear that constructive mathematics is mathematics based on intuitionistic logic, i.e. classical (usual) logic where the law of the excluded middle ($A \vee \neg A$), or equivalently the absurdity rule (suppose $\neg A$ and deduce a contradiction) or the double negation law (from $\neg\neg A$ we can derive A) aren't allowed. The idea of Heyting was to define the meaning (semantic) of formulae by the set of its proofs. This interpretation of formulae have in its sequels the rejection of the law of the excluded middle otherwise we would have a universal method for obtaining a proof of A or a proof of $\neg A$ for any proposition A. This idea, referred in the literature as BHK-interpretation [18], gives the way to link constructive mathematics to programming by the equivalence:

$$\begin{array}{rcl} \text{proof} & = \text{term} = & \text{program} \\ \text{theorem} & = \text{type} = & \text{specification} \end{array}$$

This is the Curry-Howard correspondence which leads [6], via typed lambda-calculus, to a new programming paradigm [5, 16, 20, 7]; Rather than write a program to compute a function one would instead prove a corresponding theorem and then extract the function from the proof. Examples of such systems are Nuprl [20] and Coq [16].

From the constructive mathematical point of view, as developed by Bishop [12], the algorithmic processes are usually left implicit in the proofs. This practice is more flexible but requires some work to obtain a form of the proof that is computer-readable.

3 A discrete nonstandard constructive model of $\mathbb{R}$

One of the common remarks about nonstandard analysis is that this theory is deeply nonconstructive. However, from the practical point of view, nonstandard analysis has undeniable constructive aspects. This is particularly true for the Harthong-Reeb line as Reeb himself explained [9]. In this work, we will consolidate this impression by showing that the system $\mathcal{A}_\omega$ verifies the constructive axiomatic proposed by Bridges [7]. First, let us note that, $\mathcal{A}_\omega$ comes (by construction) with a binary equality relation $=_\omega$, a binary relation $>_\omega$(greater than), a corresponding inequality relation $\neq_\omega$, two binary operations $+_\omega$ and $*_\omega$ with respectively neutral elements 0_ω and 1_ω (where $0_\omega \neq_\omega 1_\omega$) and two unary operation $-_\omega$ and $x \mapsto x^{(-1)_\omega}$. Let us note also that all the foregoing relations and operations are extensional. An important point in our treatment of the relations $=_\omega$ and $>_\omega$ is that our definitions and proofs comply with the constructive rules. Another important point is that we identify the standard integers (the elements of $\mathbb{Z}^{\text{st}}$) with the usual (constructive) integers. Nevertheless, we treat the relations $=$ and $>$ on the whole set $\mathbb{Z}$ with the usual rules of classical logic. Let us now prove that $(\mathcal{A}_\omega, +_\omega, *_\omega, =_\omega, >_\omega)$ satisfies a first group of axioms which deals with the basic algebraic properties of $\mathcal{A}_\omega$.

R1. $\mathcal{A}_\omega$ is a *Heyting field*: $\forall X, Y, Z \in \mathcal{A}_\omega$,

1. $X +_\omega Y =_\omega Y +_\omega X$,
2. $(X +_\omega Y) +_\omega Z =_\omega X +_\omega (Y +_\omega Z)$,
3. $0_\omega +_\omega X =_\omega X$,
4. $X +_\omega (-_\omega X) =_\omega 0_\omega$,
5. $X \times_\omega Y =_\omega Y \times_\omega X$,
6. $(X \times_\omega Y) \times_\omega Z =_\omega X \times_\omega (Y \times_\omega Z)$,
7. $1_\omega \times_\omega X =_\omega X$,
8. $X \times_\omega X^{(-1)_\omega} =_\omega 1_\omega$ if $X \neq_\omega 0_\omega$,
9. $X \times_\omega (Y +_\omega Z) =_\omega X \times_\omega Y +_\omega X \times_\omega Z$.

Proof. Since $+_\omega$ is the same as the classical $+$, the properties (1.), (2.), (3.) and (4.) are verified.
(5.) $X \times_\omega Y = \lfloor \frac{XY}{\omega} \rfloor = \lfloor \frac{YX}{\omega} \rfloor = Y \times_\omega X =_\omega Y \times_\omega X$.
(6.) From the definition, we get $(X \times_\omega Y) \times_\omega Z = \lfloor \lfloor \frac{X.Y}{\omega} \rfloor \frac{Z}{\omega} \rfloor$. Using several times the decomposition $U = \lfloor U \rfloor - \{U\}$ with $0 \leq \lfloor U \rfloor < 1$, we obtain

$$(X \times_\omega Y) \times_\omega Z = \left\lfloor \frac{XYZ}{\omega^2} \right\rfloor + \left\{ \frac{XYZ}{\omega^2} \right\} - \left\{ \frac{XY}{\omega} \right\} \frac{Z}{\omega} - \left\{ \left\lfloor \frac{X.Y}{\omega} \right\rfloor \frac{Z}{\omega} \right\}$$

Since $Z \in \mathcal{A}_\omega$, there is a standard $n \in \mathbb{N}$ such that $|Z| \leq n\omega$. Hence, we have

$$\left| \left\{ \frac{XYZ}{\omega^2} \right\} - \left\{ \frac{XY}{\omega} \right\} \frac{Z}{\omega} - \left\{ \left\lfloor \frac{X.Y}{\omega} \right\rfloor \frac{Z}{\omega} \right\} \right| \leq n + 2$$

and thus, $(X \times_\omega Y) \times_\omega Z) =_\omega \left\lfloor \frac{XYZ}{\omega^2} \right\rfloor$.
A similar treatment gives $X \times_\omega (Y \times_\omega Z) =_\omega \left\lfloor \frac{XYZ}{\omega^2} \right\rfloor$.
(7.) $1_\omega \times_\omega X = \omega \times_\omega X = \lfloor \frac{\omega X}{\omega} \rfloor = \lfloor X \rfloor = X =_\omega X$.
(8.) $X \times_\omega X^{(-1)_\omega} = \lfloor \frac{X \frac{\omega^2}{X}}{\omega} \rfloor = \lfloor \omega \rfloor = \omega = 1_\omega$.
(9.) The definitions lead to $X \times_\omega (Y +_\omega Z) = \lfloor \frac{X.Y + X.Z}{\omega} \rfloor$ and also to

$$X \times_\omega Y +_\omega X \times_\omega Z = \left\lfloor \frac{XY}{\omega} + \frac{XZ}{\omega} \right\rfloor + \left\{ \frac{XY}{\omega} + \frac{XZ}{\omega} \right\} - \left\{ \frac{XY}{\omega} \right\} - \left\{ \frac{XZ}{\omega} \right\}$$

Since $\left| \left\{ \frac{XY}{\omega} + \frac{XZ}{\omega} \right\} - \left\{ \frac{XY}{\omega} \right\} - \left\{ \frac{XZ}{\omega} \right\} \right| \leq 3$, we get the result. $\square$

R2. Basic properties of $>_\omega$: $\forall X, Y, Z \in \mathcal{A}_\omega$,

1. $\neg (X >_\omega Y \textit{ and } Y >_\omega X)$,
2. $(X >_\omega Y) \Rightarrow \forall Z \ (X >_\omega Z \textit{ or } Z >_\omega Y)$,
3. $\neg(X \neq_\omega Y) \Rightarrow X =_\omega Y$,
4. $(X >_\omega Y) \Rightarrow \forall Z \ (X +_\omega Z >_\omega Y +_\omega Z)$,
5. $(X >_\omega 0_\omega \textit{ and } Y >_\omega 0_\omega) \Rightarrow X \times_\omega Y >_\omega 0_\omega$.

Proof. (1.) The definition of $X >_\omega Y$ implies $X > Y$. Thus, starting with $X >_\omega Y$ and $Y >_\omega X$, we get ($X > Y$ *and* $Y > X$) which is a contradiction for the usual rules on the integers.
(2.) We know that there is a standard $n \in \mathbb{N}$ such that $n(X - Y) \geq \omega$. Thus, for $Z \in \mathcal{A}_\omega$, we get $n(X - Z) + n(Z - Y) \geq \omega$. Hence, $2n(X - Z) \geq \omega$ or $2n(Z - Y) \geq \omega$ which gives the result.
(3.) Let us recall that $\neg(X \neq_\omega Y)$ is equivalent to $\neg((X >_\omega Y) \vee (Y >_\omega X))$. We suppose that the existence of a standard $n \in \mathbb{N}$ such that $n(X - Y) \geq \omega$ or $n(Y - X) \geq \omega$ leads to a contradiction. Let $k \in \mathbb{N}$ be an arbitrary standard number; since $(k|X - Y| < \omega) \vee (k|X - Y| \geq \omega)$, we get $k|X - Y| < \omega$.
(4.) We suppose that there exists a standard $n \in \mathbb{N}$ such that $n(X - Y) \geq \omega$. Hence, for every $Z \in \mathcal{A}_\omega$ we have $n((X + Z) - (Y + Z)) \geq \omega$.
(5.) We suppose that there exists a standard $(n, m) \in \mathbb{N}^2$ such that $nX \geq \omega$ and $mY \geq \omega$. Hence, $mnX \times_\omega Y = mn\lfloor \frac{XY}{\omega} \rfloor = mn\frac{XY}{\omega} - mn\{\frac{XY}{\omega}\} \geq \omega - mn \geq \frac{\omega}{2}$. Thus, $2mnX \times_\omega Y \geq \omega$. □

Before we deal with the third group of axioms, let us just recall that we identify the constructive integers with the standard ones. As usual in a Heyting field, we embed the constructive integers in our system by the map $n \mapsto n *_\omega 1_\omega = n\omega$.

R3. Special properties of $>_\omega$:

1. **Axiom of Archimedes**: For each $X \in \mathcal{A}_\omega$ there exists a constructive $n \in \mathbb{Z}$ such that $X < n$.
2. **The constructive least-upper-bound principle**: Let S be a nonempty subset of $\mathcal{A}_\omega$ that is bounded above relative to the relation $\geq_\omega$, such that for all $\alpha, \beta \in \mathcal{A}_\omega$ with $\beta >_\omega \alpha$, either β is an upper bound of S or else there exists $s \in S$ with $s >_\omega \alpha$; then S has a least upper bound.

Proof. (1.) Since the definition of $\mathcal{A}_\omega$ is $\{x \in \mathbb{Z} \ ; \ \exists^{st} n \in \mathbb{N} \ |x| < n\omega\}$, the property R3.1. is immediately satisfied.
(2.) The pattern or our proof follows the heuristic motivation given by Bridges in [21]. We choose an element s_0 of S and an upper bound b'_0 of S in $\mathcal{A}_\omega$. Then, we consider the new upper bound $b_0 := b'_0 + 1_\omega$ of S so that $s_0 <_\omega b_0$. We define $\alpha_0 := \frac{2}{3}s_0 + \frac{1}{3}b_0$ and $\beta_0 := \frac{1}{3}s_0 + \frac{2}{3}b_0$. Since $s_0 <_\omega b_0$, we also have $\alpha_0 <_\omega \beta_0$. According to the hypothesis relative to the set S, two cases occur.

- *First case: β_0 is an upper bound of S.* Therefore we define $s_1 := s_0$ and $b_1 := \beta_0$.
- *Second case: there is $s \in S$ such that $\alpha_0 <_\omega s$.* Then, we define $s_1 := s$ and $b_1 := b_0 + s - \alpha_0$.

In each case, we get an element s_1 of S and an upper bound b_1 of S such that $\min\limits_{0 \leq k \leq 1} b_k \geq s_1 \geq s_0$ and $b_1 - s_1 =_\omega \frac{2}{3}(b_0 - s_0)$. According to the external inductive defining principle, there is an internal sequence $(s_k, b_k)_{k \in \mathbb{N}}$ in $\mathbb{Z}^2$ such that, for all standard $n \in \mathbb{N}$, we know that $s_n \in S$, b_n is an upper bound of S and

$$\min_{0 \leq k \leq n} b_k \geq s_n \geq \ldots \geq s_1 \geq s_0 \quad \text{and} \quad b_n - s_n =_\omega \left(\frac{2}{3}\right)^n (b_0 - s_0)$$

where the function min is relative to the usual order relation $\leq$ on $\mathbb{Z}$. Hence, from the overspill principle we can deduce the existence of an infinitely large number $\nu \in \mathbb{N}$, such that

$$\min_{0 \leq k \leq \nu} b_k \geq s_\nu \geq \ldots \geq s_1 \geq s_0$$

Then, we consider the element $b := \min_{0 \leq k \leq \nu} b_k$ of $\mathcal{A}_\omega$ and we want to show that b is a least upper bound of S.
- Given any element $s \in S$, we know that the property $b \geq_\omega s$ is constructively equivalent to $\neg(s >_\omega b)$. If we suppose that $s >_\omega b$, we can find a standard $n \in \mathbb{N}$ such that $s - b >_\omega b_n - s_n$. Since $b_n \geq b \geq s_n$, we have $s - b >_\omega b_n - s_n \geq b_n - b$ and thus $s - b >_\omega b_n - b$ which leads to the contradiction $s >_\omega b_n$. Hence, $b \geq_\omega s$.
- Given $b >_\omega b'$, we can choose a standard $n \in \mathbb{N}$ such that $b - b' >_\omega b_n - s_n$. Thus, we have also $b - b' > b_n - s_n$ and $b_n \geq b \geq s_n \geq b'$. As a consequence, $(b - s_n) + (s_n - b') >_\omega b_n - s_n \geq b - s_n$ so that $s_n >_\omega b'$. Hence, we have find an element s of S such that $s >_\omega b'$. □

4 Conclusion

In this paper, we have proposed a first look into a formal and constructive approach to discrete geometry based on nonstandard analysis. One of the common remarks about nonstandard analysis is that this theory is deeply nonconstructive. However, nonstandard practice has undeniable constructive aspects. This is particularly true for the Harthong-Reeb line as Reeb himself wrote [9, 11]. The arithmetization of the Euler scheme, that led to the Reveillès discrete straight line definition, is a good illustration of this. We tried to consolidate this constructive impression by showing that the system $\mathcal{A}_\omega$ verifies the constructive axiomatic proposed by Bridges [7]. The model $\mathcal{A}_\omega$ is defined with a weaker axiomatic than the one usually used for nonstandard analysis such as IST [13]. Indeed, IST allows all sorts of ideal (non constructive) objects to appear. Our weaker, restraint, axiomatic only induces a non trivial scale on the set of integers.

Of course, we are not the first to explore the relationship between constructive mathematics and nonstandard analysis. For instance, there are the deep works of Palmgren [22, 23] who introduced some new constructive approaches to nonstandard analysis. Actually, our study is completely independent of these developments, mainly because we remain within the framework of an usual axiomatic which is just a weakening of the theory IST of Nelson [13].

Our long term goal is to show that $\mathcal{A}_\omega$ represents a "good" discrete constructive model of the continuum. This work represents only the very first step towards this goal. If this succeeds, every constructive proof done in this set and based on the constructive axiomatic can be translated into an algorithm (with some work). Future works includes the production of such proofs in the discrete geometry context. We hope that it may be possible, following ideas such as the arithmetization of the Euler scheme, to compute differential properties on discrete object such as normal, curvature and so by using constructive mathematics.

References

1. Reveillès, J.P., Richard, D.: Back and forth between continuous and discrete for the working computer scientist. Annals of Mathematics and Artificial Intelligence, Mathematics and Informatic **16** (1996) 89–152
2. Harthong, J.: Éléments pour une théorie du continu. Astérisque **109/110** (1983) 235–244
3. Harthong, J.: Une théorie du continu. In Barreau, H., Harthong, J., eds.: La mathématiques non standard, Paris, Editions du CNRS (1989) 307–329
4. Bishop, E., Bridges, D.: Constructive Analysis. Springer-Verlag (1985)
5. Martin-Lof, P.: Constructive mathematics and computer programming. In Logic, Methodology and Philosophy of Science VI (1980) 153–175
6. Howard, W.A.: The formulae-as-types notion of construction. To H.B. Curry: Essays on Combinatory Logic, Lambda-calculus and Formalism (1980) 479–490
7. Bridges, D.S.: Constructive mathematics: A foundation for computable analysis. Theor. Comput. Sci. **219** (1999) 95–109
8. Mc Laughlin, W.I.: Resolving Zeno's paradoxes. Scientific American (1994) 84–89
9. Diener, F., Reeb, G.: Analyse Non Standard. Hermann, Paris (1989)
10. Diener, M.: Application du calcul de Harthong-Reeb aux routines graphiques. [11] 424–435
11. Salanski, J.M., Sinaceurs, H., eds.: Le Labyrinthe du Continu, Springer (1992)
12. Bishop, E.: Foundations of Constructive Analysis. McGraw-Hill (1967)
13. Nelson, E.: Internal set theory: A new approach to nonstandard analysis. Bulletin of the American Mathematical Society **83** (1977) 1165–1198
14. Robinson, A.: Non-standard analysis. American Elsevier, New York (1974) 2nd ed.
15. Coquand, T., Huet, G.P.: A selected bibliography on constructive mathematics, intuitionistic type theory and higher order deduction. j-J-SYMBOLIC-COMP **3** (1986) 323–328
16. Coquand, T., Huet, G.P.: The calculus of constructions. Inf. Comput. **76** (1988) 95–120
17. Beeson, M.J.: Foundations of constructive Mathematics. A Series of Modern Surveys in Mathematics. Springer (1985)
18. Troelstra, A.S., van Dalen, D.: Constructivism in Mathematics: An Introduction. Volume I. North-Holland (1988)
19. Heyting, A.: Intuitionism an introduction. Studies in Logic and Foundations of Mathematics (1956)
20. Constable, R.L., Allen, S.F., Bromley, H.M., Cleaveland, W.R., Cremer, J.F., Harper, R.W., Howe, D.J., Knoblock, T.B., Mendler, N.P., Panangaden, P., Sasaki, J.T., Smith, S.F.: Implementing Mathematics with the Nuprl Development System. Prentice-Hall (1986)
21. Bridges, D., Reeves, S.: Constructive mathematics, in theory and programming practice. Technical Report CDMTCS-068, Centre for Discrete Mathematics and Theorical Computer Science (1997)
22. Palmgren, E.: A constructive approach to nonstandard analysis. Annals of Pure and Applied Logic **73** (1995) 297–325
23. Palmgren, E.: Developments in constructive nonstandard analysis. The Bulletin of Symbolic Logic **4** (1998) 233–272