Ω-Arithmetization: a Discrete Multi-resolution Representation of Real Functions

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Abstract. Multi-resolution analysis and numerical precisions problems are very important subjects in fields like image analysis or geometrical modeling. In the continuation of previous works of the authors, we expose in this article a new method called the Ω-arithmetization. It is a process to obtain a multi-scale discretization of a continuous function that is solution of a differential equation. The constructive properties of the underlying theory leads to algorithms which can be exactly translated into functional computer programs without uncontrolled numerical error. An important part of this work is devoted to the definition and the study of the theoretical framework of the method. Some significant examples of applications are described with details.
We consider that this paper is appropriate for the Theoretical Track.
Keywords: discrete geometry, nonstandard analysis, multi-resolution analysis, constructive mathematics.

1 Introduction

In some previous works [1, 2], the authors have systematically studied a method of discretization called the arithmetization method. Let us note that the principle of this method was already at work in Reveillès's studies that led successfully to the definition of the discrete analytical line [3–5]. This arithmetization process is a way to discretize a continuous curve solution of a differential equation. The informal point of view [6, 7] that *the real line* $\mathbb{R}$ *is the same thing as the discrete line* $\mathbb{Z}$ *seen from far away* is the intuitive basis of this method. The second idea is to transform the usual approximation Euler scheme of the continuous solution into an equivalent integer scheme.

A rigorous implementation requires a model of the set $\mathbb{Z}$ of integer numbers together with a notion of infinitely large number (i. e. a scale on $\mathbb{Z}$). In the works already cited, such a model was introduced with the help of an axiomatic version of nonstandard analysis. The major imperfection of this approach is that the infinitely large integers which arise in the corresponding method have only an axiomatic status. Consequently, in the applications with concrete computations, it is impossible to give an exact numerical representation of these numbers; in such a situation, we are forced to choose sufficiently large values in an arbitrary manner[1]. Hence, this choice is only a metaphoric representation of the theoretical framework.

[1] For instance, for the figure of [2] page 2024, we took $\beta = 50$.

In the present paper, we propose to rebuild the arithmetization method on the basis of the notion of Ω-numbers introduced by Laugwitz and Schmieden [8–10]. Roughly speaking, an Ω-number (natural, integer or rational) is a sequence of numbers of same nature together with an adapted equality relation. The sets of Ω-numbers are extending the corresponding sets of usual numbers with the added advantage of providing a natural concept of infinitely large integer numbers: for instance, an Ω-integer α represented by a sequence (α_n) of integers is such that $\alpha \simeq +\infty$ if $\lim_{n\to+\infty} \alpha_n = +\infty$ in the usual meaning. Clearly, these infinite numerical entities are perfectly constructive.

After having chosen an Ω-integer ω such that $\omega \simeq +\infty$, we can define the Harthong-Reeb line $\mathcal{HR}_\omega$ which is a numerical system consisting of Ω-integers with the additional property of being roughly equivalent to the real line system. Not only the elements of $\mathcal{HR}_\omega$ have a constructivity flavor, but we can show that the structure of this system partially fits with the constructive axiomatic developed by Bridges[11].

With this, it is possible to develop the Ω-arithmetization as an arithmetization method based on this new framework. The principle of this method is unchanged and the resulting algorithm is formally the same. The new and crucial facts are the following:

- Firstly, the algorithm operates on Ω-numbers in a complete constructive way and consequently, in the applications, we can represent exactly all the entities present in the theory.
- Secondly, the result of the algorithm appears to be a discrete multi-resolution representation of the real function on which the method is applied.

From the first point, we deduce that the implementation of the method does not lead to uncontrolled approximation errors. Even for the authors, the second point was a (good) surprise.

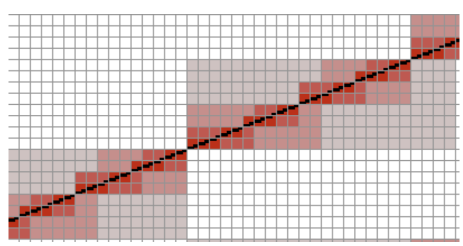

Fig. 1. Graphical representations of the multi-resolution aspects of the Ω-arithmetization of the real function $X(T) = 2T/5$. (Full explanation in section 4).

In fact, this multi-resolution aspect is a normal consequence of the Ω-arithmetization: this is in relation with the very nature of the scaling parameter β of the method (see section 4). Since β is now an infinitely large Ω-integer, it encodes an infinity of increasing scales. The arithmetization process gives simultaneously a discretization of the initial real function at each of these scales.

Since nowadays many developments in image analysis, geometrical modelling, etc. comprise multi-resolution approaches and must deal with numerical precision problems, the Ω-arithmetization is a new tool which has the interesting property of taking into account these two aspects.

The paper is organized as follow: in part 2, we introduce the Ω-numbers and study their general mathematical and logical properties, in part 3, we use the Ω-numbers to define an Harthong-Reeb line $\mathcal{HR}_\omega$ and finally, in part 4 we present the Ω-arithmetization.

2 The Ω-numbers of Laugwitz and Schmieden

In this section we will present the notion of Ω-numbers introduced by Laugwitz and Schmieden [8–10]. For the most part, we follow the presentations of these authors, but on some points, we have introduced new developments and, from our point of view, important distinctions. The Ω-numbers are nonstandard numbers but the encompassing theory has two complementary characteristics: it seems theoretically weaker than the usual versions of nonstandard analysis [12–14] but it has an undeniable flavour of constructivity suggesting the possibility of explicit and exact computations. The principal goal of Laugwitz and Schmieden was to build a new approach to real analysis based only on the introduction of a set of Ω-rational numbers which is an extension of the usual set $\mathbb{Q}$. In our case and in view of the arithmetization process, we are mainly interested in the Ω-integers but we will occasionally use the Ω-rationals.

The first step is to extend a given formal theory T (unspecified but including an elementary theory of integer and rational numbers) by introducing a new number constant Ω and a new rule (BD) described thereafter. This leads to a new theory $T\langle\Omega\rangle$ in which we can form new statements depending on Ω for which the truth is given by the following *Basic Definition* (BD):

Let $S(n)$ be a statement of T depending of $n \in \mathbb{N}$. If $S(n)$ is true for almost $n \in \mathbb{N}$, then $S(\Omega)$ is true.

We specify that here and in all this article, the expression "*almost* $n \in \mathbb{N}$" means "for all $n \in \mathbb{N}$ from some level", i.e. "$(\exists N \in \mathbb{N})$ such that $(\forall n \in \mathbb{N})$ with $n > N$". Since Ω can be substituted to any natural number, it denotes an Ω-number which is the first example of Ω-integer. Immediately, we can verify that Ω is infinitely large, i.e. greater than every elements of $\mathbb{N}$. Indeed, for $p \in \mathbb{N}$, we apply (BD) to the statement $p < n$ which is true for almost $n \in \mathbb{N}$; thus $p < \Omega$ for each $p \in \mathbb{N}$.

The second step is to describe a world of mathematical objects which is a realization of the extended theory $T\langle\Omega\rangle$. For this purpose, we consider the set of sequences of integers or rational numbers. On this set, we introduce the equivalence relation $\mathcal{R}$ such that, for $a = (a_n)$ and $b = (b_n)$[2], we have $a\mathcal{R}b$ if and only if $a_n = b_n$ for *almost* $n \in \mathbb{N}$. Then, we introduce the following definition:

$$\boxed{\textit{Each equivalence class for the relation } \mathcal{R} \textit{ is called an } \Omega\textit{-number.}}$$

In the general case, an Ω-number is also called an Ω-rational number. We agree to identify each sequence of numbers $a = (a_n)$ with the Ω-number equal to the equivalence class of a. Given a sequence $a = (a_n)$ such that $a_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$, we can say that $a = (a_n)$ is an Ω-integer. Finally, we decide that the symbol Ω is the name of the particular Ω-number $(n)_{n\in\mathbb{N}}$. The following development will show that these choices are coherent.

Let $\mathbb{Z}_\Omega$ be the set of Ω-integers, $\mathbb{N}_\Omega$ be the set of Ω-integers $c = (c_n)$ such that $c_n \geqslant 0$ for almost $n \in \mathbb{N}$ and $\mathbb{Q}_\Omega$ be the set of Ω-rational numbers. We consider the embedding $i : \mathbb{Z} \to \mathbb{Z}_\Omega$ which associates to each $p \in \mathbb{Z}$ the constant sequence of value p. An Ω-integer $a = (a_n)$ is said *standard* if a belongs to the image of the preceding embedding, i.e. if there exists $p \in \mathbb{N}$ such that $a_n = p$ for almost $n \in \mathbb{N}$. Any sequence of integers $f = (f(n))$ is a map $f : \mathbb{N} \to \mathbb{Z}$ which has a natural extension $f : \mathbb{N}_\Omega \to \mathbb{Z}_\Omega$ defined by $f(a) =_{def} (f(a_n))_{n\in\mathbb{N}}$ for $a = (a_n)$. For each Ω-integer $b = (b_n)$, we can extend the underlying sequence to $\mathbb{N}_\Omega$ and we obtain in particular $b_\Omega = (b_n) = b$. Applying this property to $(n)_{n\in\mathbb{N}}$, we find again $\Omega = (n)_{n\in\mathbb{N}}$, which partly shows the consistency of our previous choice. We do the same for the Ω-rational numbers.

Any operation or relation defined on $\mathbb{Z}$ (or $\mathbb{Q}$) naturally extends to $\mathbb{Z}_\Omega$ (or $\mathbb{Q}_\Omega$). For instance, for every Ω-numbers $a = (a_n)$ and $b = (b_n)$ let us set:

- $a + b =_{def} (a_n + b_n)_{n\in\mathbb{N}}$ and $-a =_{def} (-a_n)$ and $a \times b =_{def} (a_n \times b_n)_{n\in\mathbb{N}}$;
- $a > b =_{def} [(\exists N \forall n > N)\ a_n > b_n]$ and $a \geqslant b =_{def} [(\exists N \forall n > N)\ a_n \geqslant b_n]$;
- $|a| =_{def} (|a_n|)$.

It is easy to check that $(\mathbb{Z}_\Omega, +, \times)$ is a commutative ring with the constant sequence of value 0 as zero and the constant sequence of value 1 as unit. The previous map $i : \mathbb{Z} \to \mathbb{Z}_\Omega$ is an injective ring homomorphism which allows to identify $\mathbb{Z}$ with the subring of standard elements of $\mathbb{Z}_\Omega$. From now, we identify any integer $p \in \mathbb{Z}$ with the Ω-integer $i(p)$ equal to the sequence of constant value p.

For the implementation of an arithmetization process based on Ω-integers, we need an extension of the euclidean division to the Ω-integers and of the floor and the fractional part functions to the Ω-rational numbers.

- Given two Ω-integers $a = (a_n)$ and $b = (b_n)$ verifying $b > 0$, there is an unique $(q, r) \in \mathbb{Z}_\Omega^2$ such that $a = bq + r$ and $0 \leqslant r < b$. Indeed, since $b_n > 0$ from some level $N \in \mathbb{N}$, we can set $q = (q_n)$ and $r = (r_n)$ where, for $n \geqslant N$, q_n is the quotient of a_n by b_n and r_n is the remainder of this euclidean division, and for $n < N$ the values of q_n and r_n are arbitrary (for instance 0). We will use the usual notations $a \div b$ for the quotient q and $a \bmod b$ for the remainder r.

[2] Although this is not always indicated, in our sequences, the index n takes all the values $0, 1, \ldots$ in $\mathbb{N}$.

– Given an Ω-rational number $r = (r_n)$, there is a unique $\lfloor r \rfloor \in \mathbb{Z}_\Omega$ and a unique $\{r\} \in \mathbb{Q}_\Omega$ such that $(0 \leqslant \{r\} < 1) \wedge (r = \lfloor r \rfloor + \{r\})$. Indeed, we can choose $\lfloor r \rfloor = (\lfloor r_n \rfloor)$ and similarly $\{r\} = (\{r_n\})$.

Regarding the order relation, the usual properties true on $\mathbb{Z}$ are not always verified on $\mathbb{Z}_\Omega$. For instance

$$(\forall a, b \in \mathbb{Z}_\Omega) \quad (a \geqslant b) \vee (b \geqslant a) \tag{1}$$

is not valid as we can see for the particular Ω-integers $a = ((-1)^n)_{n\in\mathbb{N}}$ and $b = ((-1)^{n+1})_{n\in\mathbb{N}}$. Nevertheless, given two arbitrary Ω-integers $a = (a_n)$ and $b = (b_n)$, we have

$$(\forall n \in \mathbb{N}) \quad (a_n \geqslant b_n) \vee (b_n \geqslant a_n). \tag{2}$$

Using (BD), we obtain $(a_\Omega \geqslant b_\Omega) \vee (b_\Omega \geqslant a_\Omega)$ and thus (1) since $a_\Omega = a$ and $b_\Omega = b$. Hence, there is a contradiction. To avoid it, we might admit that the application of (BD) leads to a notion of truth weaker than the usual notion. Hence, we introduce an important logical distinction:

A statement is said weakly true *in case it derives from* (BD).

In contrast to the weak truth, we may use the terms of strong truth for the usual truth. For instance, (1) is weakly true but not strongly true, and the weak truth of (1) means exactly that (2) is (strongly) true. In the sequel, we will use the following properties.

Proposition 1. *The following statements are weakly true on* $\mathbb{Z}_\Omega$*:*
(1) $\forall (x, y) \in \mathbb{Z}^2_\Omega \quad (x < y) \vee (x \geq y)$*;*
(2) $\forall (x, y, z) \in \mathbb{Z}^3_\Omega \quad (x + y > z) \Rightarrow (2x > z) \vee (2y > z)$*.*

Proof. Let $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$. For each $n \in \mathbb{N}$, we have

$$(x_n < y_n) \vee (x_n \geq y_n) \text{ and } (x_n + y_n > z_n) \Rightarrow (2x_n > z_n) \vee (2y_n > z_n)$$

Thus, we can apply (BD) and we get the two statements.

Let us remark that the first statement says that the order relation on $\mathbb{Z}_\Omega$ is (weakly) decidable.

Returning to the Ω-rational numbers, we can check that $(\mathbb{Q}_\Omega, +, \times, \geqslant)$ is a commutative ordered field for the weak truth. Given two Ω-integers $a = (a_n)$ and $b = (b_n)$, if $b \neq 0$ in the weak meaning, then b has an inverse b^{-1} in $\mathbb{Q}_\Omega$ and $a/b =_{def} a \times b^{-1}$ is an Ω-rational number. Conversely, if $r \in \mathbb{Q}_\Omega$ is weakly different from 0, then there is a unique pair $(a, b) \in \mathbb{Q}^2_\Omega$ with $b > 0$ such that $r = a/b$; then, it is easy to check that we have the usual relations $\lfloor r \rfloor = a \div b$ and $\{r\} = (a \bmod b)/b$.

An Ω-rational number $a = (a_n)$ is said *limited* in case there is a standard $p \in \mathbb{N}$ such that $|a| \leqslant p$ where $|a| = (|a_n|)$; this means that $|a_n| \leqslant p$ for almost $n \in \mathbb{N}$. Let $\mathbb{Q}^{lim}_\Omega$ be the set of limited Ω-rational numbers. In the same way, we say that a is *infinitely small* and we write $a \simeq 0$ in case $p|a| \leqslant 1$ for every $p \in \mathbb{N}$.

For $a, b \in \mathbb{Q}_\Omega$, we write $a \simeq b$ when $a - b \simeq 0$ and $a \lesssim b$ when $p(a-b) \leq 1$ for every $p \in \mathbb{N}$. ; it is easy to check that $\simeq$ is an equivalence relation and that $\lesssim$ is an order relation on $\mathbb{Q}_\Omega$. This leads to the numerical system $(\mathbb{Q}_\Omega^{lim}, \simeq, \lesssim, +, \times)$ which is, for Laugwitz and Schmieden [9], an equivalent of the classical system of the real numbers $(\mathbb{R}, =, \leqslant, +, \times)$.

3 An Harthong-Reeb line based on Ω-integers

The Harthong-Reeb line is a numerical line which is, in some meaning, both discrete and continuous. For obtaining such a paradoxical space, the basic idea is to make a strong contraction on the set $\mathbb{Z}$ such that the prescribed infinitely large $\omega \in \mathbb{N}$ becomes the new unit; the result of this scaling is a line which looks like the real one. Historically, this system is at the origin of the definition of the

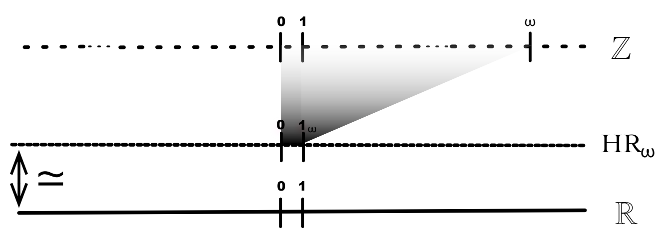

Fig. 2. An intuitive representation of the Harthong-Reeb line.

analytic discrete line proposed by J.P. Reveillès [3, 4] in discrete geometry. For a rigorous implementation of this idea, we must have a mathematical concept of infinitely large numbers. In previous works [15, 1, 2] on this subject, this was done with the help of an axiomatic version of nonstandard analysis in the spirit of Internal Set Theory [13]. Our purpose in the present section is to define an Harthong-Reeb line based on the notion of Ω-integers introduced in the previous section. Our main motivation is to obtain a more constructive version of the Harthong-Reeb line allowing an exact translation of the arithmetization process into computer programs.

Although the definition has already been stated in the previous section, we recall that an Ω-number a is infinitely large if, for all $p \in \mathbb{N}$, we have $p \leq |a|$. If a is infinitely large and $a > 0$ we note $a \simeq +\infty$. We already know that $\Omega = (n)_{n\in\mathbb{N}} \simeq +\infty$. More generally, for $a = (a_n)$, it is easy to check that $a \simeq +\infty$ if and only if $\lim_{n\to+\infty} a_n = +\infty$.

In the present and the next section, the symbol ω denotes a fixed Ω-integer such that $\omega \simeq +\infty$.

Let us remark that ω may be different from Ω. We only know that there is a sequence (ω_n) of natural numbers such that $\omega = (\omega_n)$ and $\lim_{n\to+\infty} \omega_n = +\infty$. Now, we are going to give the definition of the Harthong-Reeb line which results of the scaling on $\mathbb{Z}_\Omega$ such that ω becomes the new unit.

Definition 1. *We consider the following set*

$$\mathcal{HR}_\omega = \{x \in \mathbb{Z}_\Omega,\ \exists p \in \mathbb{N},\ |x| \leq p\,\omega\}$$

and the relations, operations and constants on $\mathcal{HR}_\omega$ described by the following definitional equalities: for all $(x, y) \in \mathcal{HR}_\omega^2$, we set

- $(x =_\omega y) =_{def} (\forall p \in \mathbb{N})\,(p|x-y| \leq \omega)$;
- $(x >_\omega y) =_{def} (\exists p \in \mathbb{N})\,(p(x-y) \geq \omega)$;
- $(x \neq_\omega y) =_{def} (x >_\omega y) \vee (x <_\omega y)$;
- $(x \leq_\omega y) =_{def} (\forall z \in \mathcal{HR}_\omega)\,(z <_\omega x \Rightarrow z <_\omega y)$;
- $(x +_\omega y) =_{def} (x + y)$ *and* $0_\omega =_{def} 0$ *and* $-_\omega\, x =_{def} -x$;
- $(x \times_\omega y) =_{def} ((x \times y) \div \omega)$ *and* $1_\omega =_{def} \omega$ *and* $x^{(-1)_\omega} =_{def} (\omega^2 \div x)$ *for* $x \neq_\omega 0$.

Then, the Harthong-Reeb line is the numerical system $(\mathcal{HR}_\omega, =_\omega, \leq_\omega, +_\omega, \times_\omega)$.

We can say that $\mathcal{HR}_\omega$ is the set of Ω-integers which are limited at the scale ω. Note the unusual way of introducing separately the two order relations and the non-equality relation; in fact, this procedure is quite traditional from a constructive point of view.

Proposition 2. *For every $x = (x_n)$ and $y = (y_n)$ in $\mathcal{HR}_\omega$, we have the following equivalences:*

(1) $x =_\omega y \Longleftrightarrow \forall p \in \mathbb{N}\ \exists M_p \in \mathbb{N}\ \forall n \geq M_p \quad p|x_n - y_n| \leq \omega_n$
(2) $x >_\omega y \Longleftrightarrow \exists p \in \mathbb{N}\ \exists M_p \in \mathbb{N}\ \forall n \geq M_p \quad p(x_n - y_n) \geq \omega_n$
(3) $x \leq_\omega y \Longleftrightarrow \forall p \in \mathbb{N},\ p(x-y) \leq \omega$

Proof. The points (1) and (2) results of the definition of the order relation $\leq$ on $\mathbb{Z}_\Omega$. We will only give the outline of a proof of (3).
Let us suppose that $x \leq_\omega y$. For every $p \in \mathbb{N} \setminus \{0\}$, we consider $z_p =_{def} x - \lfloor \omega/p \rfloor$. Since $z_p <_\omega x$, we obtain $z_p <_\omega y$. Thus, there is $k \in \mathbb{N}$ such that $k\,(y - x + \lfloor \omega/p \rfloor) \geq \omega$. Hence, for every $p \in \mathbb{N}$

$$p(x-y) \leq p\lfloor \omega/p \rfloor - p\omega/k = p\,(\omega/p - \{\omega/p\}) - p\omega/k \leq \omega$$

Let us suppose now that $p(x-y) \leq \omega$ for each $p \in \mathbb{N}$. We consider an arbitrary $z \in \mathcal{HR}_\omega$ such that $z <_\omega x$. Thus, there is $k \in \mathbb{N}$ such that $k(x-z) \geq \omega$. We obtain $k(y-z) \geq k(y-x) + \omega$ and since $2k(y-x) \geq -\omega$ we get $2k(y-z) \geq \omega$ and thus $z <_\omega y$.

Now, we want to show that the Harthong-Reeb line is equivalent to the system of real numbers. In this context, the appropriate model for the real line is the system $(\mathbb{Q}_\Omega^{lim}, \simeq, \lesssim, +, \times)$ of limited Ω-rational numbers of Laugwitz and

Schmieden described in the previous section. To this end, we introduce the two following maps:

$$\left\{ \begin{array}{rcl} \varphi_\omega : \mathcal{HR}_\omega & \to & \mathbb{Q}_\Omega^{lim} \\ x & \mapsto & x/\omega \end{array} \right\} \text{ and } \left\{ \begin{array}{rcl} \psi_\omega : \mathbb{Q}_\Omega^{lim} & \to & \mathcal{HR}_\omega \\ u & \mapsto & (\lfloor \omega u \rfloor) \end{array} \right\}$$

The proof of the following properties is straightforward.

Proposition 3. *For every $x, y \in \mathcal{HR}_\beta$ and $u \in \mathbb{Q}_\Omega^{lim}$, we have :*

- $x \leq_\omega y \quad \Rightarrow \quad \varphi_\omega(x) \lesssim \varphi_\omega(y)$;
- $\varphi_\omega(x +_\omega y) \simeq \varphi_\omega(x) + \varphi_\omega(y)$;
- $\varphi_\omega(x \times_\omega y) \simeq \varphi_\omega(x) \times \varphi_\omega(y)$;
- $\varphi_\omega(0_\omega) \simeq 0$ *and* $\varphi_\omega(1_\omega) \simeq 1$;
- $x =_\omega y \Leftrightarrow \varphi_\omega(x) \simeq \varphi_\omega(y)$;
- $\forall u \in \mathbb{Q}_\Omega^{lim}\ \exists x \in \mathcal{HR}_\omega \quad \varphi_\omega(x) \ \simeq\ u$;
- $\psi_\omega \ \circ\ \varphi_\omega(x) \ =_\omega\ x$ *and* $\varphi_\omega \circ\ \psi_\omega(u) \ \simeq\ u$.

We can summarize these properties by saying that φ_ω *is an isomorphism from* $(\mathcal{HR}_\omega, =_\omega, \leq_\omega, +_\omega, \times_\omega)$ *to* $(\mathbb{Q}_\Omega^{lim}, \simeq, \lesssim, +, \times)$ and that ψ_ω *is the inverse isomorphism.*

Since the Harthong-Reeb line $\mathcal{HR}_\omega$ is a kind of model of the real line, it is natural to wonder about the constructive content of this new numerical system. With regards to the constructivism, we only recall that these mathematics are characterized by the BHK-interpretation of the logical constants[3] and, for more precisions we refer to the excellent description given in [16]. Although we do not develop this point in this article, we have shown that the Harthong-Reeb line satisfied the axiomatic presentation of the constructive real line proposed by Bridges [16, 17]. Of course, for $\mathcal{HR}_\omega$ some of the axioms of Bridges are only weakly true. The proof is long and technical and will appear in a future paper.

4 Arithmetization with *Ω*-integers

In the previous sections, we have shown that, given an infinitely large Ω-integer ω, the corresponding Harthong-Reeb line $\mathcal{HR}_\omega$ is a relatively constructive numerical system which is roughly equivalent to the real numbers system. This equivalence gives us a way to represent continuous entities (real numbers, real function, etc.) into the discrete model $\mathcal{HR}_\omega$. Moreover, this representation comes with a strong computational content that allows to derive concrete algorithms. To illustrate this, we give the arithmetization of a linear function and an exponential function.

For this purpose, we will implement the arithmetization method presented in [2]. The new point is that we are now working with the rich structure of the

[3] The interpretation of Brouwer, Heyting and Kolmogorov which defines the intuitionistic logic.

Ω-integers. Hence, we consider a real function $X : T \mapsto X(T)$ which is solution of the Cauchy problem $X' = F(T,X)$ with the initial condition $X(A) = B$. Approximations of the function X are obtained using the Euler scheme with integration step h and real variables T_k and X_k:

$$\begin{cases} T_0 = A \ ; \ X_0 = B \\ T_{n+1} = T_n + h \\ X_{n+1} = X_n + F(T_n, X_n) \times h \end{cases} \tag{3}$$

The arithmetization method transfers the Euler approximation scheme to the discrete world $\mathcal{HR}_\omega$. To this end, we choose ω such that there is $\beta \simeq +\infty$ in $\mathbb{Z}_\Omega$ with $\omega = \beta^2$ for the product of $\mathbb{Z}_\Omega$ and we consider that $h = 1/\beta$. Since the map $\psi_\omega : U \mapsto \lfloor U\omega \rfloor$ is an equivalence between the real line and the Harthong-Reeb line $\mathcal{HR}_\omega$, it is natural to introduce in (3) the change of variables $t_k =_{def} \lfloor \omega T_k \rfloor$ and $x_k =_{def} \lfloor \omega X_k \rfloor$. Then, neglecting some terms τ such that $\tau =_\omega 0$, we get the following scheme which is an arithmetic analogue of (3) with Ω-integer variables t_k, x_k

$$\begin{cases} t_0 = a \ ; \ x_0 = b \\ t_{n+1} = t_n + \beta \\ x_{n+1} = x_n + f(t_n, x_n) \div \beta \end{cases} \tag{4}$$

where $f(t_n, x_n) = \lfloor \omega F(t_n/\omega, x_n/\omega) \rfloor$, $a = \lfloor \omega A \rfloor$ and $b = \lfloor \omega B \rfloor$. Since the integration step is now equal to $1/\beta$ with $\beta \simeq +\infty$, the discrete function whose graph is the set of the points (t_n, x_n) is an exact[4] representation of the initial real continuous function $T \mapsto X(T)$. This discrete function suffers however from a major imperfection: its domain is not a connected one at all, since $t_{n+1} - t_n = \beta \simeq +\infty$. In order to correct this defect, we perform the scaling:

$$\begin{aligned} \psi_\beta \circ \varphi_\omega : \mathcal{HR}_\omega &\longrightarrow \mathcal{HR}_\beta \\ x \quad &\longmapsto \lfloor x\omega/\beta \rfloor = x \div \beta. \end{aligned}$$

whose meaning is that we observe now the discrete function at the intermediate scale β. In order to compute the effect of this scaling, it is convenient to introduce the following notation: for every $x \in \mathcal{HR}_\omega$ we write $x = \widetilde{x}\beta + \widehat{x}$, where $\widetilde{x} =_{def} x \div \beta$ and $\widehat{x} =_{def} x \bmod \beta$. The operations $\div$ and mod are the quotient and the rest in the euclidean division of x by β in $\mathbb{Z}_\Omega$. Using this notations, we see that $\widetilde{x} \in \mathcal{HR}_\beta$ is the result of the scaling on $x \in \mathcal{HR}_\omega$. As a result, from (4) we obtain the following *Ω-arithmetization of the Euler Scheme at the intermediary scale β*

$$\begin{cases} \widetilde{t}_0 = a \div \beta, \ \widetilde{x}_0 = b \div \beta \text{ and } \widehat{x}_0 = b \bmod \beta \\ \widetilde{t}_{n+1} = \widetilde{t}_n + 1 \\ \widetilde{x}_{n+1} = \widetilde{x}_n + (\widehat{x}_n + \widetilde{f}_n) \div \beta \\ \widehat{x}_{n+1} = (\widehat{x}_n + \widetilde{f}_n) \bmod \beta \end{cases} \tag{5}$$

where $\widetilde{f}_n = f(\widetilde{t}_n\beta + a \bmod \beta, \widetilde{x}_n\beta + \widehat{x}_n) \div \beta$ and $f(t,x) = \lfloor \omega\, F(t/\omega, x/\omega) \rfloor$. Now, the relevant variables are $\widetilde{t}_k$ and $\widetilde{x}_k$ $(k = 0, 1, \dots)$ while $\widehat{x}_k$ $(k = 0, 1, \dots)$ are

[4] This discrete function contains the same information as the original continuous function

auxiliary variables that manage the approximation due to the euclidean division. The important outcome of this scaling is that the discrete function whose graph is the set of points $(\widetilde{t}_k, \widetilde{x}_k)$ is now defined over a connected domain. This function is the *arithmetization of the initial real function X at the intermediate scale β*. It is a discrete and exact representation of X.

From a practical point of view, the Ω-integers used into the algorithm associated to (5) are nothing else than a sequence of integers. Thus, implementation into a computer program requires to manage objects that are intrinsically functions from $\mathbb{N}$ to $\mathbb{Z}$. Functional programming languages are well suited to deal with such objects; our implementation uses the O'Caml language [18]. On the

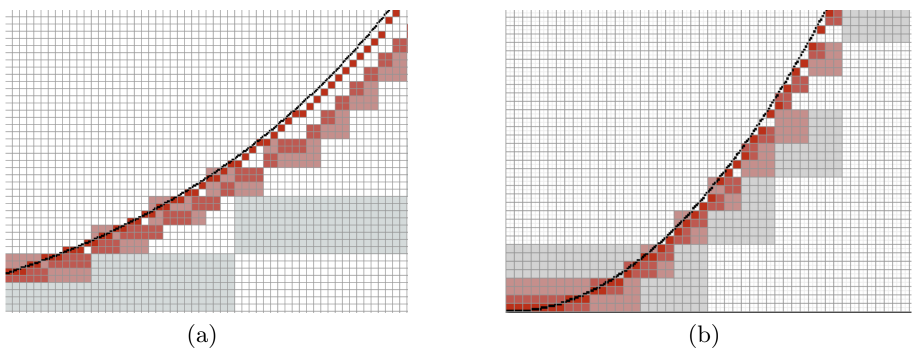

Fig. 3. Graphical representations of the Ω-arithmetization of an exponential function (a) and a parabolic function (b).

figure 3, we give graphical representations of the arithmetization of the exponential function $t \mapsto e^t/3$ and the parabolic function $t \mapsto t^2$. The parameter β is the identity map on $\mathbb{N}$. The color encodes the level n in the Ω-integers: bluish gray for $n = 5$, light pink for $n = 10$, pink for $n = 20$, red for $n = 40$ and black for $n = 200$. The different discrete graphs are reduced to the same scale, so that the pixel size is inversely proportional to the level n.

Consequently, the discretization obtained also appears to be a multi-resolution analysis. Each level or scale is represented by the colored pixel of a given size and this size is inversely proportional to the corresponding scale. This important aspect follows from the nature of the scale parameter $\beta \simeq +\infty$ as Ω-number. This number is fundamentally a sequence (β_n) of natural numbers such that $\lim_{n\to+\infty} \beta_n = +\infty$. Then, the principle of the method applied to a real function X is to compute simultaneously, for every $n \in \mathbb{N}$, a discrete approximation and a scaling of ratio β_n of this function X.

5 Conclusion

In the present paper, we have introduced the Ω-arithmetization as a method which gives a discrete and multi-scale representation of a continuous function solution of a differential equation. Due to the structure of the Ω-integers, we obtain completely constructive algorithms which can be exactly translated into functional computer programs. As a consequence, these programs do not generate any numerical error. Moreover, the result appears to be a new tool for a multi-resolution analysis of discrete functions arising from continuous ones.

In future works on this subject, we plan to study systematically this form of multi-resolution analysis and its applications to discrete geometry. In addition, we intend to change our general theoretical framework; we want to move to the formalism of constructive type theory of P. Martin-Löf [19, 20]. The first reason is that this stark approach of mathematics and computer science is well suited for both developing constructive mathematics and writing programs. Furthermore, Martin-Löf has already developed a nonstandard extension of constructive type theory [21] in which we dispose of infinitely large natural numbers. Hence, it would be possible and interesting to build a multi-resolution analysis and more generally a theory of scaling transformations in this formalism.

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