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Insight in discrete geometry and computational content of a discrete model of the continuum☆

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ABSTRACT

This article presents a synthetic and self contained presentation of the discrete model of the continuum introduced by Harthong and Reeb [J. Harthong, Éléments pour une théorie du continu, Astérisque 109/110 (1983) 235–244.[1]; J. Harthong, Une théorie du continu, in: H. Barreau, J. Harthong (Eds.), La mathématiques non standard, Éditions du CNRS, 1989, pp. 307–329.[2]] and the related arithmetization process which led Reveillès [J.-P. Reveillès, Géométrie discrète, calcul en nombres entiers et algorithmique, Ph.D. Thesis, Université Louis Pasteur, Strasbourg, France, 1991.[3]; J.-P. Reveillès, D. Richard, Back and forth between continuous and discrete for the working computer scientist, Annals of Mathematics and Artificial Intelligence, Mathematics and Informatic 16(1–4) (1996) 89–152.[4]] to the definition of a discrete analytic line. We present then some basis on constructive mathematics [E. Bishop, D. Bridges, Constructive Analysis, Springer, Berlin, 1985.[5]], its link with programming [P. Martin-Löf, Constructive mathematics and computer programming, in: Logic, Methodology and Philosophy of Science, vol. VI, 1980, pp. 153–175.[6]; W.A. Howard, The formulae-as-types notion of construction, To H.B. Curry: Essays on Combinatory Logic, Lambda-calculus and Formalism, 1980, pp. 479–490.[7]] and we propose an analysis of the computational content of the so-called Harthong–Reeb line. More precisely, we show that a suitable version of this new model of the continuum partly fits with the constructive axiomatic of $\mathbb{R}$ proposed by Bridges [Constructive mathematics: a foundation for computable analysis, Theoretical Computer Science 219(1–2) (1999) 95–109.[8]]. This is the first step of a more general program on a constructive approach of the scaling transformation from discrete to continuous space.

1. Introduction

This paper is part of a special issue of the discrete geometry in computer imagery conference [9]. The defining moment that led to the creation of this conference and to 20 years of renewed work in the field of discrete geometry has been the definition of the discrete analytical straight line by Reveillès [3]:

Given $a, b, \gamma, \tau \in \mathbb{Z}$, the discrete analytic line *with slope (a, b), with thickness τ and origin γ is the set of points $(X, Y) \in \mathbb{Z}^2$ such that*

$$\gamma \leqslant aX - bY < \gamma + \tau$$

A new approach in discrete geometry, namely discrete analytical geometry (DAG), has been a result of this formula and some of the papers in this issue of Computer & Graphics are a direct result of this work. The DAG has come a long way since these early works. Different objects such as lines, circles, planes, hyperplanes, hyperspheres, m-dimensional flats [10] have been studied. Recognition and reconstruction algorithms proposed [11]. Few people, however, remember in which context this well know formula was first proposed [9]. At the end of the eighties, at the university of Strasbourg, Reeb and Harthong developed a nonstandard model of the continuum based on integers, the discrete-continuous Harthong–Reeb model [2]. This arithmetical description of the continuum was first tested on the numerical resolution of differential equations with integer numbers. Indeed, the framework is especially well suited to solve differential problems considering the historical context. Calculus, as initiated by Leibniz and Newton, deals with the concept of infinitesimals that are very small non-zero quantities. These infinitesimals have been used to define the notion of derivatives. However, even if powerful methods were developed by Leibniz and Newton, the notion of

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infinitesimal numbers was not well defined. These numbers, that are smaller than any positive number but still different from zero, did not satisfy usual properties of real numbers. For example, any multiple of an infinitesimal number remains an infinitesimal number. This does not satisfy the Archimedean property: if x and y are two numbers such that $x<y$ then there exists an integer n such that $y<n.x$. Some paradoxes, such Zeno's paradox [12], also questioned the foundations of calculus. In the 19th century, this led to the development of what is now the classical approach to calculus, based on the notion of limits defined on the continuum of real numbers. Later, in the mid of the 20th century an alternative approach, the nonstandard analysis (NSA), was proposed which adds infinitesimals and infinitely large numbers to the real numbers.

Ironically, considering the differential origin of the DAG theory, one of the difficulties the development of discrete geometry faces today is the difficulty of correctly defining and using differential concepts. Our claim is that these difficulties come from a lack of theoretical foundations and effective methods (algorithms) to compute them. This is our motivation to reinvestigate the original NSA approach of discrete geometry. We propose, however, to extend these early works by looking at it from the constructive point of view, adding a computational context. As Georges Reeb himself noted [13,14], his model is strongly linked to the intuitionistic interpretation of classical mathematics. This has, however, never been really fully investigated although it represents a way to integrate the algorithmic paradigm to the continuum theory [15]. In this paper, our will is to take a look into a formal and constructive approach for discrete geometry using NSA. We also provide some insight in the papers of Harthong [1,2], Diener and Reeb [13], Diener [16] and Reveillès and Richard [4]. Many of these papers are hard to find and more of them are written in French.

The present article extends the paper proposed during the DGCI 2008 conference [9] by adding new examples, describing in more details the analysis of the constructive properties of the model and the general arithmetization process that was only partially presented. In Section 2, we describe the Harthong–Reeb line and the nonstandard axiomatic we need to describe the infinitely small and big numbers for our nonstandard framework. Section 3 presents the arithmetization of differential equations based on the Euler scheme. We illustrate how this works in practice with two examples. One is the arithmetization of the simple equation $y'=\alpha$, that led Reveillès to his well known discrete analytical line definition and, thus, to the discrete analytic geometry theory [4]. In Section 4, we show that a suitable version of the Harthong–Reeb model of the continuum partly fits with the constructive axiomatic of $\mathbb{R}$ proposed by Bridges [8]. Thus, this Harthong–Reeb model can be viewed as a partly constructive discrete-continuous model of the real line (called the Harthong–Reeb line). This is the first step in the project of giving theoretical and algorithmic definitions of scaling deformations like the transformation of a discrete space like $\mathbb{Z}$ into a continuous space like $\mathbb{R}$. We will then conclude in Section 5.

2. What is the Harthong–Reeb line?

In this section, we show that a theory of continuum equivalent to the real line can be developed using only integers [1,2,4,13,16]. The ground idea of the Harthong–Reeb model is to introduce a nontrivial scale on the usual set of integers in order to get a discrete form of the continuum. NSA is the paradigm that can be used to this purpose.

2.1. Bases of NSA on $\mathbb{N}$ and $\mathbb{Z}$

Even if axiomatic theories of NSA, such as IST [17], are available, we present here, in the spirit of some works of Nelson or Lutz [18,19],

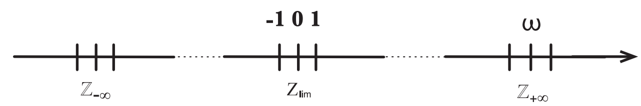

Fig. 1. Intuitive representation of the class $\mathbb{Z}_{lim}$ of the limited integers and the classes $\mathbb{Z}_{+\infty}$ and $\mathbb{Z}_{-\infty}$ of non-limited integers.

a weaker axiomatic which is well suited for our purpose. First, we introduce a new predicate *lim* over integer numbers: $lim(x)$ "means" that the integer x is limited (Fig. 1). This predicate is external to the classical integer theory and its meaning directly derives from the following axioms ANS1, ANS2, ANS3, ANS4 (and ANS5 which will be introduced later):

ANS1. *The number* 1 *is limited.*
ANS2. *The sum and the product of two limited numbers are limited.*
ANS3. *Non-limited integer numbers exist.*
ANS4. *For all* $(x,y)\in\mathbb{Z}^2$ *such that* x *is limited and* $|y|\leqslant|x|$, *the number* y *is limited.*

For reading conveniences, we introduce the following notations [9]:

- $\forall^{lim}x\ F(x)$ is an abbreviation for $\forall x\ (lim(x)\Rightarrow F(x))$ and can be read as "for all limited x, $F(x)$ stands".
- $\exists^{lim}x\ F(x)$ is an abbreviation for $\exists x\ (lim(x)\wedge F(x))$ and can be read as "exists a limited x such that $F(x)$".

Here we have to insist on the fact that these rules are added to every classical property (axioms or theorems) over integer numbers. Everything that was classically true remains true. We simply improve the language by a syntactic enrichment. These first rules imply that $\mathbb{N}$ is split into two classes, the class $\mathbb{N}_{lim}:=\{0,1,\ldots\}$ of natural limited integers (closed by arithmetical operations), and the class of natural non-limited integers (a non-limited integer is bigger than every limited integer). These non-limited integers are said to be *infinitely large*. These first axioms allow the development of an explicit and rigorous calculus on different scales.

Let us add some technical but important remarks. A mathematical property or formula which does not contain the new predicate *lim* is called an *internal* formula. For example, the formula $x+1>x$ is internal. An internal formula is, therefore, a classical formula on numbers. In contrast, an *external* formula uses explicitly the new predicate *lim*; for example, $lim(x)$ or $\forall^{lim}x, y<x$ are external formulae. Since everything that was true remains true for all internal formula $\mathscr{P}(x)$, we can build the set $P=\{x\in\mathbb{N};\mathscr{P}(x)\}$ which possesses the classic properties of subsets of $\mathbb{N}$; for example, if P is not empty and is upper bounded, then P has a maximal element. This is no longer true for an external property. For instance, if we consider the external property $lim(x)$, the class $\mathbb{N}_{lim}=\{x\in\mathbb{N};lim(x)\}$ of limited integers is a non-empty bounded part which cannot have a bigger element since $x+1$ is limited for all limited x. A class of numbers defined by an external property which cannot be a set of numbers in the classical meaning is called *external set*. Hence, $\mathbb{N}_{lim}$ is an external part of $\mathbb{N}$. Dealing with external sets that are not classical (internal) sets gives birth to a new process of demonstration called the *overspill principle*.

Proposition 1 (*Overspill principle*). *Let $\mathscr{P}(x)$ be an internal formula such that $\mathscr{P}(n)$ is true for all $n\in\mathbb{N}_{lim}$. Then, there exists an infinitely large $\nu\in\mathbb{N}$ such that $\mathscr{P}(m)$ is true for all integers m such that $0\leqslant m\leqslant\nu$.*

Proof. The class $A=\{x\in\mathbb{N};\forall y\in[0,x]\mathscr{P}(y)\}$ is an internal set (i.e. a classical set) containing $\mathbb{N}_{lim}$. Since $\mathbb{N}_{lim}$ is an external set, the inclusion $\mathbb{N}_{lim}\subset A$ is strict and leads to the result. □

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In the same way, the application of an inductive reasoning on an external formula can be illegitimate. For example, number 0 is limited, $x+1$ is limited for all limited x. Nevertheless not all integers are limited. To improve the power of our nonstandard tool, we have to add a special induction that fits with external formulae. In the following principle which is our last axiom, $\mathscr{P}$ denotes an internal or external formula:

ANS5. (*External inductive defining principle*): *We suppose that*

(1) *there is* $x_0 \in \mathbb{Z}^p$ *such that* $\mathscr{P}((x_0))$;
(2) *for all* $n \in \mathbb{N}_{lim}$ *and all sequence* $(x_k)_{0\leqslant k\leqslant n}$ *in* $\mathbb{Z}^p$ *such that* $\mathscr{P}((x_k)_{0\leqslant k\leqslant n})$ *there is* $x_{n+1} \in \mathbb{Z}^p$ *such that* $\mathscr{P}((x_k)_{0\leqslant k\leqslant n+1})$.

Therefore, there exists an internal sequence $(x_k)_{k\in\mathbb{N}}$ *in* $\mathbb{Z}^p$ *such that, for all* $n \in \mathbb{N}_{lim}$, *we have* $\mathscr{P}((x_k)_{0\leqslant k\leqslant n})$.

This principle means that the sequence of values x_k for k limited can be prolonged in an infinite sequence $(x_k)_{k\in\mathbb{N}}$ defined for all integers. Saying that this sequence is internal means that it has all the properties of the classical sequences in usual number theory. Particularly, if $\mathscr{Q}(x)$ is an internal formula, then the class $\{k \in \mathbb{N}; \mathscr{Q}(x_k)\}$ is an internal part of $\mathbb{N}$.

2.2. The system $\mathscr{HR}_\omega$

Now we are going to give the definition of the system $\mathscr{HR}_\omega$. Introduced by Diener [16], this system is the formal version of the so-called Harthong–Reeb line. In the next section, we prove that this system can be viewed as a model of the real line which is partly constructive.

Accordingly to axiom ANS3, the construction starts by considering $\omega \in \mathbb{N}$ is an infinitely large (non-limited) integer. Our purpose is to define a new numerical system such that all the elements are integers and in which ω is the new unit. Then, we introduce the underlying set of this system.

Definition 2. The set $\mathscr{HR}_\omega$ of the admissible integers considering the scale ω is defined by: $\mathscr{HR}_\omega = \{x \in \mathbb{Z}; \exists^{lim} n \in \mathbb{N} |x| < n\omega\}$.

The set $\mathscr{HR}_\omega$ is an external set. Moreover, it is an additive subgroup of $\mathbb{Z}$. We provide $\mathscr{HR}_\omega$ with the operations $+_\omega$ and $*_\omega$, the ω-scale equality, the ω-scale inequality relations (noted $=_\omega$ and $\neq_\omega$) and the order relation $>_\omega$.

We note $+, -, ., /, >, =$ the usual operations and order relation in $\mathbb{Z}$ (Fig. 2).

Definition 3. Let X and Y be any elements of $\mathscr{HR}_\omega$.

- X and Y are equal at the scale ω and we write $X=_\omega Y$ when $\forall^{lim} n \in \mathbb{N} \quad n|X-Y| \leqslant \omega$.
- Y is strictly greater than X at the scale ω and we write $Y>_\omega X$ when $\exists^{lim} n \in \mathbb{N} \quad n(Y-X) \geqslant \omega$.
- X is different from Y at the scale ω and we write $X\neq_\omega Y$ when $(X>_\omega Y$ or $Y>_\omega X)$.
- The sum of X and Y at the scale ω is $X+_\omega Y := X+Y$ (like the usual sum). For this operation, the neutral element is $0_\omega = 0$ and the opposite of each element $Z \in \mathscr{HR}_\omega$ is $-_\omega Z := -Z$.
- The product of X and Y at the scale ω is $X\times_\omega Y := \lfloor X.Y/\omega \rfloor$ (different from the usual one). The neutral element is $1_\omega := \omega$, and the inverse of each element $Z \in \mathscr{HR}_\omega$ such that $Z\neq_\omega 0_\omega$ is $Z^{(-1)_\omega} := \lfloor \omega^2/Z \rfloor$.

Let us give an informal description of $\mathscr{HR}_\omega$. It is easy to see that $X=Y$ implies $X=_\omega Y$ but that the reverse is not true. It is a little less obvious to see that we have $\forall X \in \mathscr{HR}_\omega$, $lim(X)$ implies $X=_\omega 0$ but not

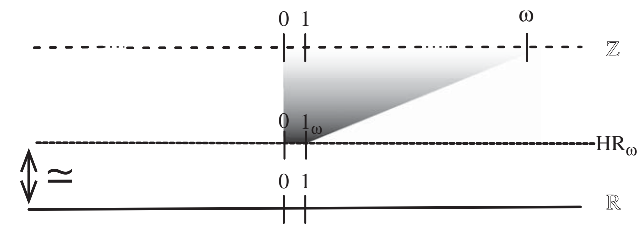


Fig. 2. An intuitive representation of $\mathscr{HR}_\omega$.

the reverse. For instance, $\lfloor\sqrt{\omega}\rfloor$ is not limited because $\omega=(\sqrt{\omega})^2$ but $\lfloor\sqrt{\omega}\rfloor =_\omega 0$ since $\forall^{lim} n \in \mathbb{N}, n.\lfloor\sqrt{\omega}\rfloor < \omega$. Moreover $\lfloor\omega\pi\rfloor$ is an element of $\mathscr{HR}_\omega$ and $\lfloor\omega\pi\rfloor =_\omega \lfloor\omega\pi\rfloor + 150 =_\omega \lfloor\omega\pi\rfloor + \lfloor\sqrt{\omega}\rfloor$. But $\omega^2 \notin \mathscr{HR}_\omega$ because there does not exist any limited integer n such that $\omega^2 < n\omega$.

Although the elements of $\mathscr{HR}_\omega$ are integers, we are going to see that the Harthong–Reeb line is equivalent to the system $(\mathbb{R}_{lim}, \simeq, \lesssim, +, \times)$ where $\mathbb{R}_{lim}$ is the set of limited real numbers, $x \simeq y$ means $(\forall^{lim} n \in \mathbb{N}\backslash\{0\} |x-y| < 1/n)$ and $x \lesssim y$ means $(x \leqslant y)$ or $(x \simeq y)$. Since, for every $X \in \mathscr{HR}_\omega$ and $x \in \mathbb{R}_{lim}$ we have $X/\omega \in \mathbb{R}_{lim}$ and $\lfloor\omega x\rfloor \in \mathscr{HR}_\omega$, we consider the two maps

$$\begin{Bmatrix} \varphi_\omega : \mathscr{HR}_\omega \to \mathbb{R}_{lim} \\ X \mapsto X/\omega \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \psi_\omega : \mathbb{R}_{lim} \to \mathscr{HR}_\omega \\ x \mapsto \lfloor\omega x\rfloor \end{Bmatrix}$$

While φ_ω is clearly additive ($\mathbb{Z}$-linear), it is not the case for ψ_ω. This is the source of some technical difficulties not always well treated in the literature.

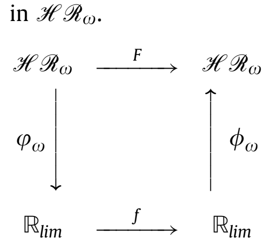
Then, for every $X, Y \in \mathscr{HR}_\omega$ and $x \in \mathbb{R}_{lim}$, we have the following properties:

- $X \leqslant_\omega Y \Rightarrow \varphi_\omega(X) \lesssim \varphi_\omega(Y)$;
- $\varphi_\omega(X +_\omega Y) \simeq \varphi_\omega(X) + \varphi_\omega(Y)$;
- $\varphi_\omega(X \times_\omega Y) \simeq \varphi_\omega(X) \times \varphi_\omega(Y)$;
- $\varphi_\omega(0_\omega) \simeq 0$ and $\varphi_\omega(1_\omega) \simeq 1$;
- $X =_\omega Y \Leftrightarrow \varphi_\omega(X) \simeq \varphi_\omega(Y)$;
- $\forall y \in \mathbb{R}_{lim} \exists X \in \mathscr{HR}_\omega \ \varphi_\omega(X) \simeq y$;
- $\psi_\omega \circ \varphi_\omega(X) =_\omega X$ and $\varphi_\omega \circ \psi_\omega(x) \simeq x$.

These properties are summarized by saying that φ_ω *is an isomorphism from* $(\mathscr{HR}_\omega, =_\omega, \leqslant_\omega, +_\omega, \times_\omega)$ *to* $(\mathbb{R}_{lim}, \simeq, \lesssim, +, \times)$ and that ψ_ω *is the inverse isomorphism*.

3. The arithmetization of the Euler scheme

In this section, we test the $\mathscr{HR}_\omega$ line as a place wherein calculus can be performed. Since $\mathscr{HR}_\omega$ is equivalent to $\mathbb{R}_{lim}$, each entity in the continuous world $\mathbb{R}_{lim}$ is represented by an equivalent one in the discrete world $\mathscr{HR}_\omega$. We call ψ_ω*-arithmetization* this process of transfer. For instance, each limited real number a is represented by $\psi_\omega(a) = \lfloor\omega a\rfloor$ in $\mathscr{HR}_\omega$. Similarly, a map $f : x \mapsto f(x)$ defined on a part of $\mathbb{R}_{lim}$ and with values in $\mathbb{R}_{lim}$ is represented by the map $F : X \mapsto F(X) := \lfloor\omega f(X/\omega)\rfloor$ defined on a part of $\mathscr{HR}_\omega$ and with values in $\mathscr{HR}_\omega$.



For a two variables map $g : (x,y) \mapsto g(x,y)$ the same process gives the discrete equivalent $G : (X,Y) \mapsto \lfloor\omega g(X/\omega, Y/\omega)\rfloor$.

The main disadvantage of this direct ψ_ω-arithmetization method is that it is based on the real numbers on which the calculations are made before coming back to $\mathscr{HR}_\omega$. On the contrary, our purpose is to stay in the discrete world of $\mathscr{HR}_\omega$ without using any real numbers. A better idea for defining a discrete equivalent of a continuous object is to find an ψ_ω-arithmetization of a construction method of this object. That is what we are going to do with the Euler scheme as a process for constructing a solution of a differential equation. This is a typical idea of Reeb who, at this time, was simply trying to make simulations of the Moiré sensing on a personal computer.

3.1. Arithmetization at the scale ω

We consider a function $x: t \mapsto x(t)$ which is a solution of a Cauchy problem

$$\begin{cases} x' = f(t,x) \\ x(a) = b \end{cases}$$

where f is a C^1 map, the constants a, b are in $\mathbb{R}_{lim}$, and the components x and t are defined on intervals in $\mathbb{R}_{lim}$. We know that we can get a good approximation of the function x by using the following Euler scheme with step $1/\beta$

$$\begin{cases} t_0 = a, \quad x_0 = b \\ t_{n+1} = t_n + 1/\beta \\ x_{n+1} = x_n + 1/\beta f(t_n, x_n) \end{cases} \tag{1}$$

The real variables t_n and x_n are such that x_n is an approximation of $x(t_n)$ and the error $|x(t_n) - x_n|$ is getting smaller when the step $1/\beta$ of the method decreases towards 0. Thus, in our context, it is interesting to consider $1/\beta \simeq 0$. Moreover, since our goal is to find an equivalent scheme in $\mathscr{HR}_\omega$ with integer variables, it is advantageous to assume that β is a divisor of ω. Finally, we suppose that

$$\exists \alpha, \beta \in \mathbb{N} \quad \omega = \alpha\beta \quad \text{and} \quad \beta \simeq +\infty \tag{2}$$

What would be a ψ_ω-arithmetization of the continuous Euler scheme (1)? It is an iterative scheme with integer variables and constants in $\mathscr{HR}_\omega$ like

$$\begin{cases} T_0 = A, \quad X_0 = B \\ T_{n+1} = T_n + C \\ X_{n+1} = X_n + \Phi(\beta, T_n, X_n) \end{cases} \tag{3}$$

where A, B, C *and the function* Φ are such that, if we go back to $\mathbb{R}_{lim}$ using the real variables $t_n := T_n/\omega$ and $x_n := X_n/\omega$, we get a scheme infinitely close to (1), that is to say, an algorithm of the form

$$\begin{cases} t_0 = a', \quad x_0 = b' \\ t_{n+1} = t_n + 1/\beta \\ x_{n+1} = x_n + 1/\beta f'(t_n, x_n) \end{cases}$$

where $a' \simeq a$, $b' \simeq b$ and $f'(t,x) \simeq f(t,x)$ for $t, x \in \mathbb{R}_{lim}$.

So let us now show how we can choose the components A, B, C and Φ of (3). It is easy for the first three terms: $A := \lfloor \omega a \rfloor$, $B := \lfloor \omega b \rfloor$ [1] and $C := \alpha$. As an arithmetical translation of the term $1/\beta\, f(t_n, x_n)$, it is quite natural to take

$$\Phi(\beta, T_n, X_n) := \lfloor (1/\beta) \lfloor \omega f(T_n/\omega, X_n/\omega) \rfloor \rfloor = F(T_n, X_n) \div \beta$$

where $F(T_n, X_n) := \lfloor \omega f(T_n/\omega, X_n/\omega) \rfloor$ is an arithmetization of $f(t_n, x_n)$ and $\div$ denotes the arithmetic operation which gives the euclidian quotient. Of course, we suppose we have already defined a ψ_ω-arithmetization F of f. Thus, we consider the following scheme:

$$\begin{cases} T_0 = A, \quad X_0 = B \\ T_{n+1} = T_n + \alpha \\ X_{n+1} = X_n + F(T_n, X_n) \div \beta \end{cases} \tag{4}$$

where $A := \lfloor \omega a \rfloor$, $B := \lfloor \omega b \rfloor$ and $F(T_n, X_n) := \lfloor \omega f(T_n/\omega, X_n/\omega) \rfloor$.

Proposition 4. *Adding the condition* $\alpha \simeq +\infty$ *to the hypothesis* (2), *the discrete scheme* (4) *is an* ψ_ω*-arithmetization of the continuous Euler scheme* (1).

Proof. Since φ_ω is additive, it is sufficient to show that $\varphi_\omega(F(T_n, X_n) \div \beta)$ may be written $1/\beta f'(t_n, x_n)$ with the condition $f'(t_n, x_n) \simeq f(t_n, x_n)$. By applying twice $\lfloor u \rfloor = u - \{u\}$, since

$$\varphi_\omega(F(T_n, X_n) \div \beta) = \frac{1}{\omega}\left(\left\lfloor \frac{\lfloor \omega f(t_n, x_n) \rfloor}{\beta} \right\rfloor \right)$$

we get

$$\varphi_\omega(F(T_n, X_n) \div \beta) = \frac{1}{\omega}\left(\frac{\omega f(t_n, x_n)}{\beta} - \frac{\{\omega f(t_n, x_n)\}}{\beta} - \left\{ \frac{\lfloor \omega f(t_n, x_n) \rfloor}{\beta} \right\} \right)$$

that is to say

$$\varphi_\omega(F(T_n, X_n) \div \beta) = \frac{1}{\beta}\left(f(t_n, x_n) - \frac{\{\omega f(t_n, x_n)\}}{\omega} - \frac{1}{\alpha}\left\{ \frac{\lfloor \omega f(t_n, x_n) \rfloor}{\beta} \right\} \right)$$

Since ω and α are infinitely large, we get the result.

3.2. Interpretation at an intermediary scale

We are now looking at the solution (T_n, X_n) of (4). It appears that this solution is a sequence of points very distant from each other because $T_{n+1} - T_n = \alpha \simeq +\infty$. In order to get the points closer together for this sequence, we can observe the solution at an intermediate scale.

We have already two intermediate scales: α and β such that $\omega = \alpha\beta$. Each of these scales provide simplifications in various lines of the scheme. One way of having more simplifications is to suppose that $\alpha = \beta$. Thus, we make a further assumption which is compatible with the previous ones (2)

$$\exists \beta \in \mathbb{N} \quad \omega = \beta^2 \quad \text{and} \quad \beta \simeq +\infty \tag{5}$$

More formally, an element $X \in \mathscr{HR}_\omega$ is interpreted to an intermediary scale β as an element of $\mathscr{HR}_\beta$ by the map

$$\psi_\beta \circ \varphi_\omega : \mathscr{HR}_\omega \longrightarrow \mathscr{HR}_\beta$$
$$X \longmapsto X \div \beta$$

In order to get the interpretation of scheme (4) at the intermediate scale β we introduce the following notation: for every $X \in \mathscr{HR}_\omega$

$$X = \widetilde{X}\beta + \widehat{X} \tag{6}$$

where $\widetilde{X} := X \div \beta$ and $\widehat{X} := X \bmod \beta$, respectively denote the quotient and the remainder in the Euclidean division of X by β. Using decomposition (6) for each component, we get *the* ψ_ω*-arithmetization of Euler scheme (1) computed at the scale* $\omega = \beta^2$ *and interpreted at the intermediary scale* β:

$$\begin{cases} \widetilde{T}_0 = A \div \beta, \quad \widetilde{X}_0 = B \div \beta \quad \text{and} \quad \widehat{X}_0 = B \bmod \beta \\ \widetilde{T}_{n+1} = \widetilde{T}_n + 1 \\ \widetilde{X}_{n+1} = \widetilde{X}_n + (\widehat{X}_n + \widetilde{F}_n) \div \beta \\ \widehat{X}_{n+1} = (\widehat{X}_n + \widetilde{F}_n) \bmod \beta \end{cases} \tag{7}$$

[1] Of course, we suppose that these two elements are already known without any calculus in $\mathbb{R}$. This knowledge may be the result of a preceding arithmetization process.

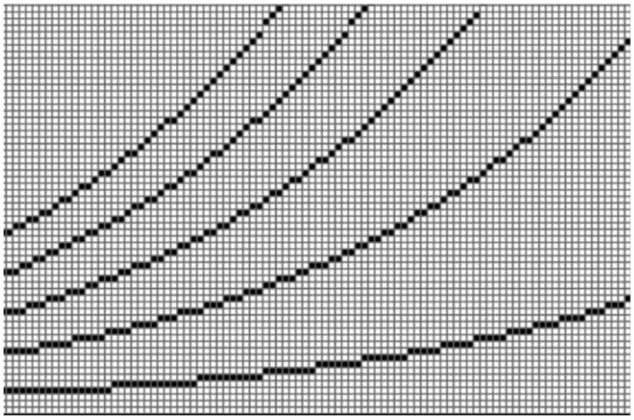


Fig. 3. The arithmetization of $t \mapsto \gamma e^t$ computed at the scale β^2 and interpreted at the scale β for $\beta = 50$ and $\gamma = 3/200, 9/200, 15/200, 21/200, 27/200$.

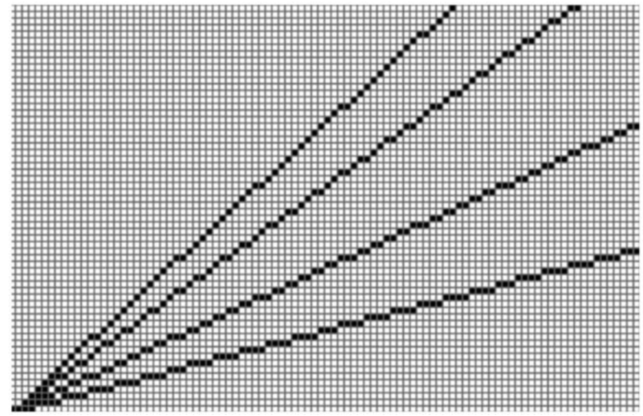


Fig. 4. The ψ_ω-arithmetization of the line $15y = ax$ computed at the scale β^2 and interpreted at the scale β for $\beta = 50$ and $a = 4, 7, 11, 14$.

where $A = \lfloor \beta^2 a \rfloor$, $B = \lfloor \beta^2 b \rfloor$ and

$$\widetilde{F}_n = F(\widetilde{T}_n\beta + A \bmod \beta, \widetilde{X}_n\beta + \widehat{X}_n) \div \beta$$
$$= \left\lfloor \omega f\left(\frac{\widetilde{T}_n\beta + A \bmod \beta}{\omega}, \frac{\widetilde{X}_n\beta + \widehat{X}_n}{\omega}\right)\right\rfloor \div \beta.$$

In scheme (7), we have to understand that the $\widetilde{T}_k$ and the $\widetilde{X}_k$ are the relevant variables and that the $\widehat{X}_k$ are auxiliary variables. Now, the set of pairs $(\widetilde{T}_k, \widetilde{X}_k)$ is the graph of a discrete function $T \mapsto X(T)$ defined on an interval $\mathbb{I}$ of $\mathbb{Z}$. We will say that this function is *the ψ_ω-arithmetization of the initial function $t \mapsto x(t)$, computed at the scale β^2 and interpreted at the scale β.*

3.3. Examples of arithmetization

3.3.1. The arithmetization of the exponential function (Fig. 3)

The exponential function $x \mapsto \gamma e^x$ is the solution of the following Cauchy problem:

$$\begin{cases} x' = x \\ x(0) = \gamma \end{cases}$$

The function f of the general theory is now the projection $(t,x) \mapsto x$. Thus, we get

$$F(\widetilde{T}_n\beta + A \bmod \beta, \widetilde{X}_n\beta + \widehat{X}_n) = \widetilde{X}_n\beta + \widehat{X}_n \quad \text{so } \widetilde{F}_n = \widetilde{X}_n$$

Since the initial condition is $x(0) = \gamma$, we have $A = 0$ and $B = \lfloor \gamma\beta^2 \rfloor$. Consequently, the arithmetization of the corresponding Euler scheme computed at the scale β^2 and interpreted at the scale β is the following:

$$\begin{cases} \widetilde{T}_0 = 0, \quad \widetilde{X}_0 = \lfloor \gamma\beta^2 \rfloor \div \beta \quad \text{and} \quad \widehat{X}_0 = 0 \\ \widetilde{T}_{n+1} = \widetilde{T}_n + 1 \\ \widetilde{X}_{n+1} = \widetilde{X}_n + (\widetilde{X}_n + \widehat{X}_n) \div \beta \\ \widehat{X}_{n+1} = (\widetilde{X}_n + \widehat{X}_n) \bmod \beta \end{cases} \tag{8}$$

This is precisely the kind of algorithm proposed by Reeb. This algorithm defines a discrete function $T \mapsto E(T)$ for $T \geqslant 0$ in $\mathscr{HR}_\beta$ which is an arithmetization of the exponential function: $E(T)/\beta \simeq e^{T/\beta}$ for $T \in \mathscr{HR}_\beta$ and $e^t \simeq E(\lfloor \beta t \rfloor)/\beta$ for $t \in \mathbb{R}_{lim}$.

3.3.2. The origin of the discrete analytical Reveillès straight line (Fig. 4)

As pointed out in the Introduction, the previous point of view leads to the definition of the well known discrete analytical straight line defined by Reveillès [3,4]. The discrete analytical straight line of Reveillès is a fundamental object that led to 20 years of renewed research in the discrete geometry community. We now show how this definition is a consequence of the ψ_ω-arithmetization of the Euler scheme applied to the elementary function $t \mapsto ct + d(\star)$.

The ψ_ω-arithmetization of the continuous Euler scheme corresponding to $(\star)$, computed at the scale β^2 and interpreted at the scale β, is

$$\begin{cases} \widetilde{T}_0 = 0, \quad \widetilde{X}_0 = \lfloor \beta^2 d \rfloor \div \beta \quad \text{and} \quad \widehat{X}_0 = \lfloor \beta^2 d \rfloor \bmod \beta \\ \widetilde{T}_{n+1} = \widetilde{T}_n + 1 \\ \widetilde{X}_{n+1} = \widetilde{X}_n + (\widehat{X}_n + K) \div \beta \\ \widehat{X}_{n+1} = (\widehat{X}_n + K) \bmod \beta \end{cases} \tag{9}$$

where K is the integer $\lfloor \beta^2 c \rfloor \div \beta \in \mathscr{HR}_\beta$. The set of pairs $(\widetilde{T}_k, \widetilde{X}_k)$ is the graph of a discrete function $T \mapsto X(T)$ defined on an interval $\mathbb{I}$ of $\mathbb{Z}$.

Proposition 5. *For all $T \in \mathbb{I}$, we have $X(T) = \lfloor C_\beta T + D_\beta \rfloor$ where $C_\beta := (\lfloor \beta^2 c \rfloor \div \beta)/\beta$ and $D_\beta := \lfloor \beta^2 d \rfloor / \beta$.*

Proof. We introduce the variable $X_n := \widetilde{X}_n\beta + \widehat{X}_n$ with values in $\mathscr{HR}_\omega$ where $\omega = \beta^2$. Then, we have $\widetilde{X}_n = X_n \div \beta$ and we get the inductive relation $X_{n+1} = X_n + K$. Thus, for each n we have $X_n = X_0 + nK$. Since $K = \lfloor \omega c \rfloor \div \beta$ and

$$X_0 = \widetilde{X}_0\beta + \widehat{X}_0 = (\lfloor \omega d \rfloor \div \beta)\beta + \lfloor \omega d \rfloor \bmod \beta = \lfloor \omega d \rfloor$$

we get the expression $X_n = n(\lfloor \omega c \rfloor \div \beta) + \lfloor \omega d \rfloor$ and finally

$$\widetilde{X}_n = \left\lfloor \frac{n(\lfloor \omega c \rfloor \div \beta) + \lfloor \omega d \rfloor}{\beta} \right\rfloor$$

Since $\widetilde{T}_n = n$, the proof is done. $\square$

The elements of the graph of the function $T \mapsto X(T)$ are the points with integer coordinates which are on or just below the continuous straight line Δ_β with equation $x = C_\beta t + D_\beta$. The slope of Δ_β is $C_\beta = (\lfloor \beta^2 c \rfloor \div \beta)/\beta \simeq c$ and for $t = 0$ we have $x = D_\beta = \lfloor \beta^2 d \rfloor / \beta \simeq \beta d$. Thus, for $d = 0$ the line Δ_β is infinitely close to the initial line.

Finally, it is easy to check that the graph of the function $T \mapsto \lfloor C_\beta T + D_\beta \rfloor$ is the set of $(T, X) \in \mathbb{Z}^2$ such that

$$0 \leqslant KT - \beta X + \beta D_\beta \leqslant \beta$$

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This kind of inequalities is at the origin of the concept of analytic discrete line introduced by Reveillès:

Given $a, b, \gamma, \tau \in \mathbb{Z}$, *the* discrete analytic line *with slope* (a, b), *with thickness* τ *and origin* γ *is the set of points* $(X, Y) \in \mathbb{Z}^2$ *such that*

$$\gamma \leqslant aX - bY < \gamma + \tau$$

4. The constructive part of the Harthong–Reeb line

In the previous section, we introduced the Harthong–Reeb line $\mathscr{HR}_\omega$, we described the ψ_ω-arithmetization process and the link with discrete analytic geometry. Up to now, the algorithmic part of this theory of the continuum was neglected. With the purpose of warranting the correctness of the algorithms developed in this framework, we are going to study its computational context.

4.1. Constructive mathematics, proofs and programs

In this part, we will briefly introduce constructive mathematics and shortly draw the links with programming. For interested readers more details can be found in [6,8,20–22].

As explained by Martin-Löf in [6]:

> The difference between constructive mathematics and programming does not concern the primitive notions […] they are essentially the same, but lies in the programmer's insistence that his programs be written in a formal notation […] whereas, in constructive mathematics […] the computational procedures are normally left implicit in the proofs […].

Constructive mathematics find their origins, at the beginning of 20th century, with the criticisms of the formalist mathematical point of view developed by Hilbert which led to what we now call "classical mathematics". Brouwer was the most radical opponent to formal mathematics in which one can prove the existence of a mathematical object without providing a way (an algorithm) to construct it [23]. But his metaphysical approach to constructivism (intuitionism) was not successful. Around 1930, the first who tried to define an axiomatization of constructive mathematics was Arend Heyting, a student of Brouwer. In the mid of the fifties, he published a treaty [24] were intuitionism is presented to both mathematicians and logicians. From Heyting's works it became clear that constructive mathematics is intuitionistic-logic based mathematics, i.e. classical (usual) logic where the law of the excluded middle ($A \vee \neg A$), or equivalently the *Reductio ad absurdium* (suppose $\neg A$ and deduce a contradiction) or the double negation law (from $\neg\neg A$ we can derive A) are not allowed. The idea of Heyting was to define the meaning (semantic) of formulae by the set of its proofs. This interpretation of formulae have in its sequels the rejection of the law of the excluded middle otherwise we would have a universal method for obtaining a proof of A or a proof of $\neg A$ for any proposition A. This idea, referred in the literature as BHK-interpretation [23], gives the way to link constructive mathematics with programming by the equivalence:

proof = term = program

theorem = type = specification

This is the Curry–Howard correspondence which leads [7], via typed lambda-calculus, to a new programming paradigm [6,8,21,25]; rather than write a program to compute a function one would instead prove a corresponding theorem and then extract the function from the proof. Examples of such systems are Nuprl [25] and Coq [21].

From the constructive mathematical point of view, as developed by Bishop [15], the algorithmic processes are usually left implicit in the proofs. This practice is more flexible but requires some work to obtain a form of the proof that is computer-readable.

One of the common remarks about NSA is that this theory is deeply nonconstructive. However, from the practical point of view, NSA has undeniable constructive aspects. This is particularly true for the Harthong–Reeb line as Reeb himself explained [13]. The arithmetization of the Euler scheme, that led to the Reveillès discrete straight line definition, is a good illustration of this aspect. In this work, we will consolidate this impression by showing that the system $\mathscr{HR}_\omega$ verifies the constructive axiomatic proposed by Bridges and Reeves [8,26].

4.2. A special logical framework for $\mathscr{HR}_\omega$

From the definitions stated in the description of $\mathscr{HR}_\omega$, we notice that a decision about the relations $>_\omega$ and $=_\omega$ needs a kind of infinite search in the whole class of limited natural numbers. Consequently, there is a great difference about the derived properties in our system depending on whether we follow classical or intuitionistic logic. Actually, we will use the following rules about the integers and the underlying logic of our reasoning:

LR1. For each $X \in \mathbb{Z}$, we have $(0 > X)$ or $(X = 0)$ or $(X > 0)$, even if X is not limited.
LR2. The limited integers are identified with the usual constructive integers.
LR3. We only use intuitionistic logic.

As a consequence, for each $X, Y \in \mathscr{HR}_\omega$ we immediately see that

(i) $(X \neq_\omega Y) \Longrightarrow \neg(X =_\omega Y)$,
(ii) $(X \neq_\omega Y) \Longleftrightarrow (\exists^{lim} n \in \mathbb{N} \quad n|X - Y| \geqslant \omega)$,

but the reverse of (i) is not true in general; similarly, for every $X, Y \in \mathscr{HR}_\omega$ we have not in general $(X >_\omega Y) \vee (X =_\omega Y) \vee (Y >_\omega X)$. Following [8,26], we define the relation $\geqslant_\omega$ only in term of $>_\omega$.

Definition 6. Let us consider $X, Y \in \mathscr{HR}_\omega$. Then $X \geqslant_\omega Y$ if and only if

$$\forall Z \in \mathscr{HR}_\omega (Y >_\omega Z \Longrightarrow X >_\omega Z)$$

Proposition 7. *Let us consider* $X, Y \in \mathscr{HR}_\omega$. *Then,*

$$((X >_\omega Y) \vee (X =_\omega Y)) \Longrightarrow X \geqslant_\omega Y$$

Proof. Let $Z \in \mathscr{HR}_\omega$ such that $Y >_\omega Z$: $\exists^{lim} n \in \mathbb{N} \quad n(Y - Z) \geqslant \omega$.

- If $X >_\omega Y$, then $\exists^{lim} m \in \mathbb{N} \quad m(X - Y) \geqslant \omega$. Then, $p(X - Z) \geqslant \omega$ for $p := \max(m, n)$ and consequently $X >_\omega Z$.
- If $X =_\omega Y$, then $\forall^{lim} m \in \mathbb{N} \quad \omega \geqslant m(X - Y) \geqslant -\omega$. Hence

$$2n(X - Z) = 2n(X - Y) + 2n(Y - Z) \geqslant -\omega + 2\omega = \omega. \quad \square$$

Again, the reverse implication is not true in general. Nevertheless:

Proposition 8. *For every* $X, Y \in \mathscr{HR}_\omega$, *the following conditions are equivalent*:

(1) $X \geqslant_\omega Y$;
(2) $(X \geqslant Y) \vee (X =_\omega Y)$;
(3) $\forall^{lim} n \in \mathbb{N} \quad \omega \geqslant n(Y - X)$.

Proof. We suppose (1). If $X \geqslant Y$, we have nothing to do. Thus, we suppose that $Y > X$. For each limited $p \in \mathbb{N}$, we consider $Z_p := Y - \lfloor \omega/p \rfloor$ in $\mathscr{HR}_\omega$. We have $Y - Z_p \geqslant \omega/p - 1$ and then $(p+1)(Y - Z_p) \geqslant (p+1)\omega/p - (p+1) \geqslant \omega$ because $\omega \geqslant p(p+1)$. Hence $Y >_\omega Z_p$ from which we get $X >_\omega Z_p$ and also $X > Z_p$. Consequently, $Y > X > Z_p$ and, for each limited $n \in \mathbb{N}$, we have

$$n|X-Y| = n(Y-X) \leqslant n(Y-Z_p) \leqslant n\lfloor \omega/p \rfloor \leqslant \frac{n}{p}\omega.$$

Choosing $p=n$, we get $n|X-Y| \leqslant \omega$ for every limited n.

- We suppose (2). If $X \geqslant Y$, then for every limited $n \in \mathbb{N}$, we have $\omega \geqslant 0 \geqslant n(Y-X)$. If $Y > X$ then $X =_\omega Y$ and for every limited $n \in \mathbb{N}$, we have $n(Y-X) = n|Y-X| \leqslant \omega$.
- We suppose (3) and we consider $Z \in \mathscr{R}_\omega$ such that $Z <_\omega Y$. Hence, there is a limited $k \in \mathbb{N}$ such that $k(Y-Z) \geqslant \omega$. Since $n(Y-X) \leqslant \omega$ for every limited $n \in \mathbb{N}$, we have

  $$X - Z = (X-Y) + (Y-Z) \geqslant -\frac{\omega}{n} + \frac{\omega}{k}.$$

  Choosing $n = 2k$, we get $2k(Y-Z) \geqslant \omega$. □

4.3. $\mathscr{HR}_\omega$ satisfies the axiomatic of bridges

In [8,26], Bridges introduced an abstract structure which is a constructive axiomatic presentation of the real line. Let us call a *Bridges–Heyting ordered field* any system which satisfies these axioms. The main result of our work is the following theorem:

Theorem 9. $(\mathscr{HR}_\omega, +_\omega, \times_\omega, =_\omega, >_\omega)$ *is a Bridges–Heyting ordered field.*

Since we only use the logical rules (LR1), (LR2) and (LR3), this result shows that *the Harthong–Reeb line is partially constructive in a non-trivial way*.[2] To prove it, we show that the system $(\mathscr{HR}_\omega, +_\omega, \times_\omega, =_\omega, >_\omega)$ satisfies the three groups of axioms (R1), (R2) and (R3) defined by Bridges.

R1. $\mathscr{HR}_\omega$ *is a Heyting field*: $\forall X, Y, Z \in \mathscr{HR}_\omega$,

(1) $X +_\omega Y =_\omega Y +_\omega X$.
(2) $(X +_\omega Y) +_\omega Z =_\omega X +_\omega (Y +_\omega Z)$.
(3) $0_\omega +_\omega X =_\omega X$.
(4) $X +_\omega (-_\omega X) =_\omega 0_\omega$.
(5) $X \times_\omega Y =_\omega Y \times_\omega X$.
(6) $(X \times_\omega Y) \times_\omega Z =_\omega X \times_\omega (Y \times_\omega Z)$.
(7) $1_\omega \times_\omega X =_\omega X$.
(8) $X \times_\omega X^{(-1)_\omega} =_\omega 1_\omega$ if $X \neq_\omega 0_\omega$.
(9) $X \times_\omega (Y +_\omega Z) =_\omega X \times_\omega Y +_\omega X \times_\omega Z$.

Proof. Since $+_\omega$ is the same as the classical $+$, the properties (1), (2), (3) and (4) are verified.

(5) $X \times_\omega Y = \lfloor XY/\omega \rfloor = \lfloor YX/\omega \rfloor = Y \times_\omega X =_\omega Y \times_\omega X$.
(6) From the definition, we get $(X \times_\omega Y) \times_\omega Z = \lfloor \lfloor X.Y/\omega \rfloor Z/\omega \rfloor$. Using several times the decomposition $U = \lfloor U \rfloor - \{U\}$ with $0 \leqslant \lfloor U \rfloor < 1$, we obtain

$$(X \times_\omega Y) \times_\omega Z = \left\lfloor \frac{XYZ}{\omega^2} \right\rfloor + \left\{ \frac{XYZ}{\omega^2} \right\} - \left\{ \frac{XY}{\omega} \right\} \frac{Z}{\omega} - \left\{ \left\lfloor \frac{X.Y}{\omega} \right\rfloor \frac{Z}{\omega} \right\}$$

Since $Z \in \mathscr{HR}_\omega$, there is a limited $n \in \mathbb{N}$ such that $|Z| \leqslant n\omega$. Hence, we have

$$\left| \left\{ \frac{XYZ}{\omega^2} \right\} - \left\{ \frac{XY}{\omega} \right\} \frac{Z}{\omega} - \left\{ \left\lfloor \frac{X.Y}{\omega} \right\rfloor \frac{Z}{\omega} \right\} \right| \leqslant n+2$$

and thus, $(X \times_\omega Y) \times_\omega Z) =_\omega \lfloor XYZ/\omega^2 \rfloor$. A similar treatment gives $X \times_\omega (Y \times_\omega Z) =_\omega \lfloor XYZ/\omega^2 \rfloor$.
(7) $1_\omega \times_\omega X = \omega \times_\omega X = \lfloor \omega X/\omega \rfloor = \lfloor X \rfloor = X =_\omega X$.
(8) $X \times_\omega X^{(-1)_\omega} = \lfloor X(\omega^2/X)/\omega \rfloor = \lfloor \omega \rfloor = \omega = 1_\omega$.
(9) The definitions lead to

$$X \times_\omega (Y +_\omega Z) = \left\lfloor \frac{X.Y + X.Z}{\omega} \right\rfloor$$

and also to

$$X \times_\omega Y +_\omega X \times_\omega Z = \left\lfloor \frac{XY}{\omega} + \frac{XZ}{\omega} \right\rfloor + \left\{ \frac{XY}{\omega} + \frac{XZ}{\omega} \right\} - \left\{ \frac{XY}{\omega} \right\} - \left\{ \frac{XZ}{\omega} \right\}$$

Since $|\{XY/\omega + XZ/\omega\} - \{XY/\omega\} - \{XZ/\omega\}| \leqslant 3$, we get the result. □

R2. *Basic properties of* $>_\omega$: $\forall X, Y, Z \in \mathscr{HR}_\omega$,

(1) $\neg(X >_\omega Y \wedge Y >_\omega X)$
(2) $(X >_\omega Y) \Rightarrow \forall Z (X >_\omega Z \text{ or } Z >_\omega Y)$
(3) $\neg(X \neq_\omega Y) \Rightarrow X =_\omega Y$
(4) $(X >_\omega Y) \Rightarrow \forall Z (X +_\omega Z >_\omega Y +_\omega Z)$
(5) $(X >_\omega 0_\omega \wedge Y >_\omega 0_\omega) \Rightarrow X \times_\omega Y >_\omega 0_\omega$

Proof. (1) The definition of $X >_\omega Y$ implies $X > Y$. Thus, starting with $X >_\omega Y$ and $Y >_\omega X$, we get ($X > Y$ and $Y > X$) which is a contradiction for the usual rules on the integers.

(2) We know that there is a limited $n \in \mathbb{N}$ such that $n(X-Y) \geqslant \omega$. Thus, for $Z \in \mathscr{HR}_\omega$, we get $n(X-Z) + n(Z-Y) \geqslant \omega$. Hence, $2n(X-Z) \geqslant \omega$ or $2n(Z-Y) \geqslant \omega$.[3]

(3) Let us recall that $\neg(X \neq_\omega Y)$ is equivalent to $\neg((X >_\omega Y) \vee (Y >_\omega X))$. We suppose that the existence of a limited $n \in \mathbb{N}$ such that $n(X-Y) \geqslant \omega$ or $n(Y-X) \geqslant \omega$ leads to a contradiction. Let $k \in \mathbb{N}$ be an arbitrary limited number; since $(k|X-Y| < \omega) \vee (k|X-Y| \geqslant \omega)$, we get $k|X-Y| < \omega$.

(4) We suppose that there exists a limited $n \in \mathbb{N}$ such that $n(X-Y) \geqslant \omega$. Hence, for every $Z \in \mathscr{HR}_\omega$ we have $n((X+Z)-(Y+Z)) \geqslant \omega$.

(5) We suppose that there are limited $n, m \in \mathbb{N}$ such that $nX \geqslant \omega$ and $mY \geqslant \omega$. Hence, $mnX \times_\omega Y = mn\lfloor XY/\omega \rfloor = mnXY/\omega - mn\{XY/\omega\} \geqslant \omega - mn \geqslant \omega/2$. Thus, $2mnX \times_\omega Y \geqslant \omega$. □

Before dealing with the third group of axioms, let us recall that we identify the constructive integers with the limited ones. As usual in a Heyting field, we embed the constructive integers in our system by the map $n \mapsto n \times_\omega 1_\omega = n\omega$ of $\mathscr{HR}_\omega$. A subset S of $\mathscr{HR}_\omega$ is the collection of elements of $\mathscr{HR}_\omega$ which satisfies a given property defined in the system. This property may be internal or external. Such a subset S is *bounded above relative to the relation* $\geqslant_\omega$ if there is $b \in \mathscr{HR}_\omega$ such that $b \geqslant_\omega s$ for all $s \in S$; the element b is called an *upper bound* of S. A *least upper bound* for S is an element $b \in \mathscr{HR}_\omega$ such that

- $\forall s \in S \; b \geqslant_\omega s$ (b is an upper bound of S);
- $\forall b' (b >_\omega b') \Rightarrow (\exists s \in S \quad s >_\omega b')$.

A least upper bound is unique: if b and c are two least upper bounds of S, then we have $\neg(b >_\omega c)$ and $\neg(c >_\omega b)$; thus, according to the properties[4] of the relations $>_\omega$, $\geqslant_\omega$ and $=\omega$, we get $c \geqslant_\omega b$ and $b \geqslant_\omega c$ and then $b =_\omega c$.

[2] For instance, our proof shows that the system $\mathscr{HR}_\omega$ does not satisfy Bridges's axioms in the same trivial way than the classical set $\mathbb{R}$ of real numbers.

[3] If k, l and m are integers such that $k + l \geqslant m$, then $2\max(k,l) \geqslant m$. Since this is only a question of sign of integers, this property is decidable in our framework.

[4] These properties are not completely trivial in intuitionistic logic.

R3. *Special properties of* $>_\omega$:

(1) *Axiom of Archimedes*: For each $X \in \mathscr{HR}_\omega$ there exists a constructive $n \in \mathbb{Z}$ such that $X < n$.
(2) *The constructive least-upper-bound principle*: Let S be a non-empty subset of $\mathscr{HR}_\omega$ that is bounded above relative to the relation $\geqslant_\omega$, such that for all $\alpha, \beta \in \mathscr{HR}_\omega$ with $\beta >_\omega \alpha$, either β is an upper bound of S or else there exists $s \in S$ with $s >_\omega \alpha$; then S has a least upper bound.

Proof. (1) Since $\mathscr{HR}_\omega = \{X \in \mathbb{Z}; \exists^{lim} n \in \mathbb{N} |X| < n\omega\}$, the first point is clear.

(2) The pattern or our proof follows the heuristic motivation given by Bridges in [26]. We choose an element s_0 of S and an upper bound b'_0 of S in $\mathscr{HR}_\omega$. Then, we consider the new upper bound $b_0 := b'_0 + 1_\omega$ of S so that $s_0 <_\omega b_0$. We define $\alpha_0 := \frac{2}{3}s_0 + \frac{1}{3}b_0$ and $\beta_0 := \frac{1}{3}s_0 + \frac{2}{3}b_0$. Since $s_0 <_\omega b_0$, we also have $\alpha_0 <_\omega \beta_0$. According to the hypothesis relative to the set S, two cases occur.

- *First case*: β_0 *is an upper bound of S*. Therefore, we define $s_1 := s_0$ and $b_1 := \beta_0$.
- *Second case*: *there is* $s \in S$ *such that* $\alpha_0 <_\omega s$. Then, we define $s_1 := s$ and $b_1 := b_0 + s - \alpha_0$. [5]

In each case, we get an element s_1 of S and an upper bound b_1 of S such that $\min_{0 \leqslant k \leqslant 1} b_k \geqslant s_1 \geqslant s_0$ and $b_1 - s_1 =_\omega \frac{2}{3}(b_0 - s_0)$. According to the external inductive defining principle, there is an internal sequence $(s_k, b_k)_{k \in \mathbb{N}}$ in $\mathbb{Z}^2$ such that, for all limited $n \in \mathbb{N}$, we know that $s_n \in S$, b_n is an upper bound of S and

$$\min_{0 \leqslant k \leqslant n} b_k \geqslant s_n \geqslant \cdots \geqslant s_1 \geqslant s_0 \quad \text{and} \quad b_n - s_n =_\omega \left(\frac{2}{3}\right)^n (b_0 - s_0)$$

where the function min is relative to the usual order relation $\leqslant$ on $\mathbb{Z}$. Hence, from the overspill principle we can deduce the existence of an infinitely large number $\nu \in \mathbb{N}$, such that

$$\min_{0 \leqslant k \leqslant \nu} b_k \geqslant s_\nu \geqslant \cdots \geqslant s_1 \geqslant s_0$$

Then, we consider the element $b := \min_{0 \leqslant k \leqslant \nu} b_k$ of $\mathscr{HR}_\omega$ and we want to show that b is a least upper bound of S.

- Given any element $s \in S$, we know that the property $b \geqslant_\omega s$ is constructively equivalent to $\neg(s >_\omega b)$. If we suppose that $s >_\omega b$, we can find a limited $n \in \mathbb{N}$ such that $s - b >_\omega b_n - s_n$. Since $b_n \geqslant b \geqslant s_n$, we have $s - b >_\omega b_n - s_n \geqslant b_n - b$ and thus $s - b >_\omega b_n - b$ which leads to the contradiction $s >_\omega b_n$. Hence, $b \geqslant_\omega s$.
- Given $b >_\omega b'$, we can choose a limited $n \in \mathbb{N}$ such that $b - b' >_\omega b_n - s_n$. Thus, we have also $b - b' > b_n - s_n$ and $b_n \geqslant b \geqslant s_n \geqslant b'$. As a consequence, $(b - s_n) + (s_n - b') >_\omega b_n - s_n \geqslant b - s_n$ so that $s_n >_\omega b'$. Hence, we have find an element s of S such that $s >_\omega b'$. □

5. Conclusion

In this paper, we recalled the origins of the discrete analytical geometry developed by Reveillès. An important tool of this approach is the use of the Harthong–Reeb line, $\mathscr{HR}_\omega$, which is a nonstandard model of the continuum based on integers. In the present work, we introduced a suitable version of $\mathscr{HR}_\omega$ which is characterized by the use of a partly intuitionistic logic and a weak axiomatic version of NSA. Then, we showed that this system $\mathscr{HR}_\omega$ satisfies the axiomatic of a constructive real line defined by Bridges. As a consequence, we can say that $\mathscr{HR}_\omega$ is constructive to some extent.

On the one hand, this property is surprising because NSA is generally thought as deeply non-constructive. According to this view, the typical nonstandard entities (like the nonstandard numbers) are basically non-constructive, as fictitious as non-measurable set of Lebesgue theory or as the axiom of choice of set theory. Consequently, theses entities would be nothing else that some non-essential artefacts of our formalism [27,28].

On the other hand, this result reflects the constructive aspects of the practice of NSA. Harthong and Reeb explain in [29] that, far from being an artefact, NSA necessarily results of an intuitionistic interpretation of the classical mathematical formalism (see also [30]). Moreover, some strange structural similarities have been noted between nonstandard and constructive proofs [31]. Finally, thanks to the works of Martin-Löf [32], Moerdijk [33] and Palmgren [34,35], there are now new presentations of NSA which completely fit with the constructive constraints. Actually, our study is completely independent of these last developments, mainly because we remain within the framework of an usual axiomatic which is just a weakening of the theory IST of Nelson [36].

Regarding to the Harthong–Reeb line $\mathscr{HR}_\omega$, we believe that the constructive content of this system is measured by the constructive quality of our logical framework defined by the three rules (LR1), (LR2) and (LR3). In this connection, the rule (LR3) is perfect and the rule (LR2) is not problematic. Conversely, the lack of constructivity of this system should also be read in this special logical context: it is clear that the combination of the rules (LR1) and (LR2) is not completely satisfactory because it is only for the usual integers that the relation $=$ and $>$ are unquestionably decidable. The weakness of our point of view may also be localized in the purely axiomatic introduction of the infinitely large numbers like ω which do not have a specific computational content.

As a consequence, our system $\mathscr{HR}_\omega$ is not completely constructive and we are far from being able to bring any proof from our system into a concrete program. In other works yet to come, we plan to investigate some more constructive versions of the Harthong–Reeb line. Our long term goal is to get a “correct” discrete and constructive model of the continuum. To this end, we intend to use the idea of Laugwitz and Schmieden on infinitely large natural numbers [37–39] and the framework of Martin-Löf type theory [6,32,40]. If we succeed, we hope that it may be possible, following ideas such as the arithmetization of the Euler scheme, to shed a new light on some differential notions in discrete geometry. The present work presents only the very first step towards this goal.

[5] In classical logic, it would be better to put $b_1 := b_0$ and then to introduce an equality test between s_1 and b_1; but this test is not constructively valid.

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